

The convergence Newton polygon of a p -adic differential equation I : Affinoid domains of the Berkovich affine line

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ABSTRACT

We prove the *finiteness* of the radius of convergence function \mathcal{R}^M of a differential module M over an affinoid domain X of the Berkovich affine line $\mathbb{A}_K^{1,\text{an}}$. This means that there exists a finite graph $\Gamma(\mathcal{R}^M) \subset X$, together with a canonical retraction $\delta_{\mathcal{R}^M} : X \rightarrow \Gamma(\mathcal{R}^M)$, such that the function $\mathcal{R}^M : X \rightarrow \mathbb{R}_{>0}$ factorizes through $\delta_{\mathcal{R}^M}$. More generally, for each $\xi \in X$, we define the *convergence Newton polygon* $NP^{\text{conv}}(M, \xi)$ of M , whose first slope is the logarithm of $\mathcal{R}^M(\xi)$, and the other slopes are the logarithms of the radii \mathcal{R}_i^M of convergence of all the Taylor solutions of M at ξ . We prove the finiteness of all the slopes $\mathcal{R}_i^M(\xi)$ of $NP^{\text{conv}}(M, \xi)$, and of its partial heights $H_i^M(\xi)$, as functions on X , together with their fundamental properties. Roughly speaking this result implies that there are only a *finite* number of numerical invariants that one can extract from the slopes of \mathcal{R}_i^M and H_i^M along the branches of X . As a corollary we have their continuity.

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Introduction

In the ultrametric context the (one variable) radius of convergence function of a differential module M is an important invariant by isomorphisms. It is a function defined over a certain Berkovich space X and its slopes along the branches of X are *numerical invariants* (by isomorphism) of M . If M is a differential module over $K((T))$, where K is trivially valued and of characteristic 0, then from the knowledge of the radius of convergence function of M (and of its submodules) one can recover the B.Malgrange *irregularity* of M , the *Poincaré-Katz rank* of M , and more generally *the entire formal Newton polygon*. The Radius of convergence function is also a major tool in the proof of the p -adic local monodromy theorem (cf. [And02], [Meb02], [Ked04]), and more recently of the Sabbah's conjectures (cf. [Ked10a]). The radius of convergence function is today one of the most

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important invariants of an ultrametric differential module. In this paper we prove its *finiteness*. Roughly speaking this implies that the numerical invariants that one can extract from the slopes of the radius of convergence function along the branches of X are *finite* in number. We now explain what this means, and we give an idea of the proof.

Let $(K, |\cdot|)$ be a complete valued ultrametric field of characteristic 0.¹ Let X be a connected affinoid domain of the Berkovich affine line $\mathbb{A}_K^{1,\text{an}}$ over K , and let $\mathcal{O}(X)$ be the K -affinoid algebra of its global sections. It is known that X is always the quotient by $\text{Gal}(K^{\text{alg}}/K)$ of a *standard set* (cf. [Ber90, 4.2]). Roughly speaking X is obtained from a closed disk $D^+(c_0, R_0)$ by removing a finite number of open disks (cf. section 1.1). In the whole paper a coordinate T of $\mathbb{A}_K^{1,\text{an}}$ is chosen. This determines the size of the disks as well as the choice of the derivation d/dT . The radius of convergence function will depend on this choice. For all $\xi \in X$ one has a canonical path $\lambda_\xi : [0, R_0] \rightarrow X$ with initial point ξ , and with end point the point ξ_{c_0, R_0} of the Shilov boundary of X defined by $D^+(c_0, R_0)$ (cf. section 1.3, [Ber90, 1.4.4]). A (*closed*) *branch* of X is the image of such λ_ξ , in fact X is the union of such branches and it has the structure of a so called *polyhedron* (cf. [Ber90, 4.1]). Moreover if Γ is a finite union of closed branches, then the inclusion $\Gamma \subset X$ admits a canonical retraction $\delta_\Gamma : X \rightarrow \Gamma$ (cf. section 1.4), and X is the topological projective limit of such retractions (cf. [BR10, Thm.2.20]). It is natural to ask whether a given function $\mathcal{R} : X \rightarrow \mathcal{T}$, where \mathcal{T} is a set, factorizes through such a retraction δ_Γ . The first point of this paper is to associate to \mathcal{R} a canonical (possibly not finite) union of branches $\Gamma(\mathcal{R})$, called the *constancy skeleton* (or simply *skeleton*) of \mathcal{R} , on which \mathcal{R} factorizes under convenient assumptions (cf. section 2). Roughly speaking $\Gamma(\mathcal{R})$ is the complement in X of the union of all the disks on which \mathcal{R} is constant. If $\Gamma(\mathcal{R})$ is a finite union of closed branches we say that \mathcal{R} is *finite* or that it has a *finite skeleton*. Next we provide a sufficient set of conditions that guarantee that \mathcal{R} is finite, continuous, and it factorizes through $\delta_{\Gamma(\mathcal{R})}$. In order to have an idea we list a rough version of them here (cf. section 2.3.1, Thm. 2.14 for a more accurate statement):

- (C1) For all $\xi \in X$ one has $\rho_{\mathcal{R}}(\xi) > 0$.
- (C2) \mathcal{R} is piecewise linear, continuous, with a finite number of breaks on each closed branch of X .
- (C3) There exists a finite union of closed branches Γ such that if $D^-(t, \rho) \cap \Gamma = \emptyset$, then \mathcal{R} is log-concave (hence decreasing by (C1)) on the branches inside $D^-(t, \rho)$.
- (C4) The modulus of all possible *non zero* slopes of \mathcal{R} at any point is lower bounded by a positive real number $\nu_{\mathcal{R}} > 0$, which is independent on the Berkovich point.
- (C5) $\Gamma(\mathcal{R})$ is *directionally finite* at all its bifurcation points i.e. there are a finite number of branches of $\Gamma(\mathcal{R})$ passing through a bifurcation point ξ of $\Gamma(\mathcal{R})$.
- (C6) \mathcal{R} is *super-harmonic* outside a finite set $\mathcal{C}(\mathcal{R}) \subseteq X$ (cf. Def. 2.10).

Among the functions satisfying these properties there are the functions of $\mathcal{O}(X)$, but also those of the type $\min(|f_1|^{-\alpha_1}, \dots, |f_n|^{-\alpha_n})$, with $\alpha_i > 0$, and many others (compare with (4.2)). These properties are modeled on those satisfied by the partial height of the Newton polygon of a differential operator (cf. section 4.1). The rough idea of the proof is that the super-harmonicity implies that at each bifurcation point of $\Gamma(\mathcal{R})$ the function \mathcal{R} has a break, while the assumption (C2) provides that there are a finite number of breaks, and hence a finite number of bifurcation points.

¹The case of $(K, |\cdot|)$ trivially valued is allowed along the whole paper (up to section 6). Indeed the convergence polygon NP^{conv} is invariant by scalar extension of K , so we can extend K to a non trivially valued base field Ω/K , apply the theorems over Ω , and then re-descend to K as explained in section 2.4. If K is trivially valued the ring of formal power series $K[[T]]$ coincides with the ring of bounded analytic functions over $D^-(0, 1)$, its fraction field $K((T))$ coincides with the ring of bounded analytic functions over $\{|T| \in]0, 1[$. One has in this case the classical theory of formal differential equations.

Let now (M, ∇) be a differential module over the differential ring $(\mathcal{O}(X), \frac{d}{dT})$. Let $Y' = G(T) \cdot Y$, $G \in M_r(\mathcal{O}(X))$, be the differential equation associated to M in a basis. One is allowed to consider Taylor solutions of this equation and test their radius of convergence at each point of $X(\Omega)$, for all complete valued field extension Ω/K . This fact permits to associate to *any* Berkovich point $\xi \in X$ a radius of convergence by testing Taylors solutions at $t_\xi := T(\xi) \in X(\mathcal{H}(\xi))$.² Namely denote by $Y(T, t_\xi)$ the Taylor solution of this equation around t_ξ , with initial value $Y(t_\xi, t_\xi) := \text{Id}$. If $Y^{(n)} = G_n(T) \cdot Y$ is the n -th iterate of the equation, then $Y(T, t_\xi) := \sum_{n \geq 0} G_n(t_\xi) \frac{(T-t_\xi)^n}{n!}$. The minimum of the radii of convergence at t_ξ of the entries of $Y(T, t_\xi)$ is given by $\mathcal{R}^Y(\xi) := \liminf_n | \frac{G_n(t_\xi)}{n!} |_\Omega^{-1/n}$. One obtains a function $\mathcal{R}^Y : X \rightarrow \mathbb{R}_{>0}$ depending on the chosen basis of M . In order to make this number invariant by base changes in M one sets

$$\mathcal{R}^M(\xi) := \min(\liminf_n \xi(G_n/n!)^{-1/n}, \rho_{\xi, X}) , \quad (0.1)$$

where $\rho_{\xi, X}$ is the radius of the largest open disk centered at $t_\xi \in X(\mathcal{H}(\xi))$ contained in $X \hat{\otimes} \mathcal{H}(\xi)$. $\mathcal{R}^M : X \rightarrow \mathbb{R}_{>0}$ is called the *radius of convergence function* of M . It represents the smallest radius of convergence of a Taylor solution of M around t_ξ . We now refine this construction by taking in account the other radii. The vector space of germs of convergent solutions at $t_\xi \in X(\mathcal{H}(\xi))$ is naturally filtered by the radius of convergence of its elements. We associate a polygon $NP^{\text{conv}}(M, \xi)$ to this filtration, called *convergence polygon of M at ξ* (cf. section 4.3). Its first slope $s_1^M(\xi) = h_1^M(\xi)$ is equal to $\ln(\mathcal{R}^M(\xi))$. For $i = 1, \dots, r$ its i -th slope is given by $s_i^M(\xi) := \ln(\mathcal{R}_i^M(\xi))$, where $\mathcal{R}_i^M(\xi) \leq \rho_{\xi, X}$ is the radius of the largest open disk centered at t_ξ on which M admits at least $r - i + 1$ linearly independent Taylor solutions, where r is the rank of M . This defines univocally $NP^{\text{conv}}(M, \xi)$ as the epigraph³ of the convex function $h : [0, r] \rightarrow \mathbb{R}$ defined by the fact that $h(0) = 0$, and that $h(\xi)$ is linear on $[i - 1, i]$ with slope $s_i^M(\xi)$. The values $h_i^M(\xi) := h(i)$ are called the i -th *partial heights*. The main result of this paper (cf. Thm. 4.7) provides important properties on the behavior of $NP^{\text{conv}}(M, \xi)$ as a function of ξ . Namely we proves that the functions $\mathcal{R}^Y, \mathcal{R}^M, s_i^M, h_i^M : X \rightarrow \mathbb{R}_{>0}$ are all *finite functions* i.e. they have a finite skeleton and factorize through it. As a consequence one has their continuity. We precise moreover a family of formal properties enjoyed by them as the piecewise linearity, convexity, super-harmonicity, integrality. Roughly speaking this result means that there are a *finite number* of numerical invariants of M that one can extract from the slopes of $\mathcal{R}^Y, \mathcal{R}^M, s_i^M, h_i^M$ along the branches of X , and that these functions are all *definable* in the sens of [LH10]. The proof is an induction on $i = 1, \dots, r$, and the first step consists in proving the finiteness of $\exp(h_1^M) = \mathcal{R}^M$. The aforementioned criterion works for $h_1^M = \mathcal{R}^M$ with respect to Γ equal to the skeleton of X , and $\mathcal{C}(\mathcal{R}^M)$ being the Shilov boundary. For $i \geq 2$ it holds for h_i^M with respect to $\Gamma := \cup_{j=1}^{i-1} \Gamma(h_j^M)$, and $\mathcal{C}(h_i^M)$ being a certain finite set depending on h_1^M, \dots, h_{i-1}^M . In order to prove the six properties (C1)–(C6) we compare the *convergence polygon* $NP^{\text{conv}}(M, \xi)$ with two other polygons: the *spectral Newton polygon* $NP^{\text{sp}}(M, \xi)$, and the *spectral Newton polygon* $NP^{\text{sp}}(\mathcal{L}, \xi)$ of a differential operator $\mathcal{L} := (\frac{d}{dT})^r + \sum_{i=0}^{r-1} g_{r-i}(T)(\frac{d}{dT})^i$ defining M in a convenient cyclic basis.

²The original idea (and one of the most fruitful one) of considering “*generic points*” is due to Bernard Dwork. In his language a *generic point* for ξ is a $t \in X(\Omega)$ satisfying $\xi(f) = |f(t)|_\Omega$ for all $f \in \mathcal{O}(X)$ (cf. section 1.2), where Ω is a large unspecified complete valued field extension of K . Considering such *generic points* have been for long time a common practice (cf. [Dwo74], [Rob75], [CD94], ...), and is still a “*routine*” by the specialists (cf. [CM02], [Meb02], ...). As a matter of facts in the papers of Dwork and Robba X is always considered as the functor associating to Ω the set $X(\Omega)$ considered as a metric subspace of $(\Omega, |\cdot|)$. Indeed a very large Ω (making all the points of X Ω -rational) is often fixed once for all, in order to work with an individual metric space $X(\Omega)$. Although unnecessary, in order to make the link between the two worlds it is convenient to systematically practice the “*yoga*” of functor of points defined by X . In fact a point $t \in X(\Omega)$ provides a bounded character $\mathcal{O}(X) \rightarrow \Omega$ of $\mathcal{O}(X)$, by $T \mapsto t$. Eventually, by the description given in [Ber90, 1.2.2,ii)], this amounts to consider another description of X itself (cf. section 1.0.1). The reader knowing the language of Berkovich will not have any problem in recognizing the usual underlying objects of Berkovich theory.

³i.e. the set of points of \mathbb{R}^2 on or above the graph of $h(\xi)$.

The slopes and the partial heights of $NP^{\text{sp}}(\mathcal{L}, \xi)$ are explicitly given in terms of the coefficients $g_i \in \mathcal{O}(X)$, and are hence finite. Now $NP^{\text{sp}}(M, \xi)$ is obtained from $NP^{\text{conv}}(M, \xi)$ “by truncation” of the large slopes (cf. section 3.2 and (4.10), see also [Ked10b, Notes of Ch.9, p.166]), while a classical result due to Young [You92] proves that the “small” slopes of these three polygons coincide (see [CM02, Thm.6.2] i.e. Prop.4.3 and Thm.5.1). In the non p -adic case, this is enough to control all the slopes since they are always “small”. In the p -adic case the “big” values of the slopes are reduced to the “small” values by using the Frobenius push-forward techniques as in [Ked10b] and [CD94].

We conclude this introduction by discussing the different definitions of the radii. The first slope of $NP^{\text{sp}}(M, \xi)$ is the logarithm of the so called *spectral radius*

$$\mathcal{R}^{M, \text{sp}}(\xi) := \min(\liminf_n \xi(G_n/n!)^{-1/n}, r(\xi)) . \quad (0.2)$$

where $r(\xi)$ is the *generic radius of ξ* .⁴ The name of this function is due to the fact that $\mathcal{R}^{M, \text{sp}}(\xi) = \omega/\|\nabla\|_{\text{sp}, \xi}$, where $\|\nabla\|_{\text{sp}, \xi}$ is the spectral norm⁵ of the connection of M with respect to the norm ξ (cf. (3.16)). $\mathcal{R}^{M, \text{sp}}(\xi)$ is the *spectral radius* studied in the whole literature (cf. [Ked10b, Def.9.4.4], [CD94, Section 2.3], [CM02], ...). $\mathcal{R}^{M, \text{sp}}$ is for certain reasons a better function than \mathcal{R}^M since it only depends on the restriction of M to $\mathcal{H}(\xi)$, and it is hence invariant by restriction to a sub-affinoid. For this reason it is much intrinsic than \mathcal{R}^M . Expressing the radius in terms of spectral norm permits to make the theory much more algebraic and hence easier to generalize in more variables (cf. [Ked10a], [KX10], ...). Unfortunately the fact that $\mathcal{R}^{M, \text{sp}}(\xi) = 0$ for all \widehat{K}^{alg} -rational point ξ implies that $\mathcal{R}^{M, \text{sp}}$ is not continuous, in fact if the valuation of K is not trivial the set of \widehat{K}^{alg} -rational points is dense in X . Moreover its skeleton $\Gamma(\mathcal{R}^{M, \text{sp}})$ is always equal to the whole space X , and $NP^{\text{sp}}(M, \xi)$ is not invariant under (non algebraic) scalar extensions of K . On the other hand \mathcal{R}^M is not stable by restriction to a sub-affinoid, mainly because of the presence of $\rho_{\xi, X}$. Hence it can not easily be glued to give a function on a general curve. This has been recently done by F.Baldassarri [Bal10] where the definition of \mathcal{R}^M has been widely improved in order to make it much more intrinsic i.e. independent on the choice of the coordinate T . We notice that Baldassarri’s definition still preserves the dependence on a chosen skeleton, and it seems that *a completely intrinsic definition does not exist*. The definition of \mathcal{R}^M adopted here is due to F.Baldassarri himself and L.Di Vizio [BV07]. And the definitions of [Bal10] reduces to ours in this more elementary case (cf. section 8). As already mentioned a direct corollary of the finiteness is the continuity of $\mathcal{R}^Y, \mathcal{R}^M, s_i^M, h_i^M : X \rightarrow \mathbb{R}_{>0}$. For $i = 1$, i.e. for $\mathcal{R}^M = \exp(h_1^M)$, this is the one-variable case of [BV07], and a special case of [Bal10]. The proof of the continuity presented here is different in nature from those of [BV07], [Bal10], and [CD94]. All of them use a result of Dwork and Robba [DR80] providing an effective growth condition on the coefficients of the generic Taylor solution. Our proof does not use it since the continuity follows from the finiteness, and eventually from the continuity of the Newton polygon of \mathcal{L} .

REMARK 0.1. *Jérôme Poineau and Amaury Thuillier (independently) pointed out that if one works with over-convergent coefficients, the continuity of \mathcal{R}^M on X (but also on curves) is a consequence of the super-harmonicity. The proof generalizes the fact that a concave function on an open interval is continuous, and is based on the explicit expression (0.1). This method seems to fail for \mathcal{R}_i^M since it can not be expressed globally on X as in (0.1), but only outside $\cup_{j=1}^{i-1} \Gamma(\mathcal{R}_j^M)$ (cf. section 7).*

Notes. A first proof of the harmonicity properties is due to P.Robba [Rob84] and [Rob85] for rank one differential equations with rational coefficients. He obtained the harmonicity of the radius

⁴This number is denoted by $\text{diam}(\xi)$ in [BR10, Section 1.4]. It is often called the *radius of the point* (cf. [Ber90, 4.2]). In the theory of differential equation $r(\xi)$ is the radius of the Dwork’s generic disk, this is why we call it generic radius.

⁵The spectral norm $\|\nabla\|_{\text{sp}, \xi}$ is defined only if ξ is a norm, in particular if $r(\xi) > 0$. We extend this definition to the points satisfying $r(\xi) = 0$ by setting $\|\nabla\|_{\text{sp}, \xi} := +\infty$. In this way $\mathcal{R}^{M, \text{sp}}(\xi) = \omega/\|\nabla\|_{\text{sp}, \xi}$ for all $\xi \in X$.

function by expressing its slopes by means of the index (cf. [Rob84, Thm. 4.2, p.201]), and then deducing the harmonicity from the additivity of the indexes (cf. [Rob84, Prop.4.5, p.207]).⁶

For differential modules over an annulus the super-harmonicity properties along the skeleton of the annulus are obtained in [Ked10b] that have been for us a constant source of inspiration. This paper is intended to generalize [Ked10b, Thm. 11.3.2] to the context of Berkovich spaces.

The existence of the skeleton of the radius of convergence function found his genesis in [Bal10] where F.Baldassarri announced a forthcoming paper with the proof of the finiteness of the individual function \mathcal{R}^M in the more general framework of Berkovich curves.

A first proof of the finiteness of a function appears in [Ked10a, Section 5] in the case of a Berkovich closed unit disk over $K := k((z))$, where k is a trivially valuated field. The definitions of [Ked10a] are given *ad hoc* to deal with a closed disk and there are discrepancies with those of this paper, especially for the definition of the skeleton of a function (which is defined in [Ked10a] in term of the slopes). It turns out that the two definitions eventually coincide over a closed disk, and in fact certain techniques of this paper are not far from those of [Ked10a] and [Ked10b].

A proof of the finiteness of the radius function have been obtained by G.Christol [Chr11] for differential equations of rank one with polynomial coefficients. The proof uses an explicit formula for the radius function that we have contributed to realize (cf. the introduction of [Chr11]). The generalization of such a formula to rank one differential equation with arbitrary coefficients is the object of a forthcoming paper. This have been the starting point of the present paper.

The complete proof of the continuity and finiteness of the convergence Newton polygon over a quasi-smooth K -analytic Berkovich curve is obtained in the sequel of this paper [PP12].

Structure of the paper. In sections 1 we provide notations, and in section 2 we introduce the skeleton $\Gamma(\mathcal{R})$ together with the aforementioned finiteness criterion (cf. Thm. 2.14). In sections 3 and 4 we define all the polygons and we state the main result (cf. Thm. 4.7). In sections 5 and 6 we adapt to our context some results, and in section 7 we give the proof of 4.7.

1. Notation

All rings are commutative with unit element. \mathbb{R} is the field of real numbers, and $\mathbb{R}_{\geq 0} := \{r \in \mathbb{R} \mid r \geq 0\}$. For all field L we denote its algebraic closure by L^{alg} , by \widehat{L} its completion (if it has a meaning), by $L[T]$ the ring of polynomial with coefficients in L , and by $L(T)$ the fraction field of $L[T]$. In this paper $(K, |\cdot|)$ will be a complete field of characteristic 0 with respect to an ultrametric absolute value $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ i.e. verifying $|1| = 1$, $|a \cdot b| = |a||b|$, and $|a + b| \leq \max(|a|, |b|)$ for all $a, b \in K$, and $|a| = 0$ if and only if $a = 0$. We denote by $|K| := \{r \in \mathbb{R}_{\geq 0} \text{ such that } r = |t| \text{ with } t \in K\}$. Define $E(K)$ as the category of complete valued ultrametric field $(\Omega, |\cdot|_{\Omega})$ together with an isometric inclusion $e_{\Omega} : (K, |\cdot|) \rightarrow (\Omega, |\cdot|_{\Omega})$. A morphism $\Omega \rightarrow \Omega'$ in $E(K)$ is an isometric morphism of rings inducing the identity on K . The category $E(K)$ is filtering in the sense that for all $\Omega, \Omega' \in E(K)$ there exists $\Omega'' \in E(K)$ together with two morphisms $\Omega \subseteq \Omega''$ and $\Omega' \subseteq \Omega''$.

1.0.1 Berkovich spaces. An ultrametric Banach ring $(\mathcal{A}, \|\cdot\|)$ is a ring \mathcal{A} together with a norm $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ verifying for all $a, b \in \mathcal{A}$ (i) $\|a + b\| \leq \max(\|a\|, \|b\|)$, (ii) $\|ab\| \leq \|a\|\|b\|$, (iii) $\|1\| = 1$, (iv) $\|a\| = 0$ if and only if $a = 0$. A *bounded, multiplicative* semi-norm of \mathcal{A} is a map $\xi : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$ satisfying (i) $\xi(a + b) \leq \max(\xi(a), \xi(b))$, (ii) $\xi(ab) = \xi(a)\xi(b)$, (iii) $\xi(1) = 1$, (iv) $\xi(0) = 0$, (v) $\xi \leq C\|\cdot\|$ for a constant $C > 0$. By definition the Berkovich space $\mathcal{M}(\mathcal{A})$ of \mathcal{A} is space of all *bounded multiplicative semi-norm* on \mathcal{A} , and its topology is the weakest one making

⁶The principle is used in [Rob85, top of p.50] to construct the so called Robba's exponentials.

continuous each map $\mathcal{M}(\mathcal{A}) \rightarrow \mathbb{R}_{\geq 0}$ of the form $\xi \mapsto \xi(a)$. With this topology $\mathcal{M}(\mathcal{A})$ is a non empty compact Hausdorff topological space (cf. [Ber90]). If $\xi \in \mathcal{M}(\mathcal{A})$, the kernel of ξ is a prime ideal, and we denote by $\mathcal{H}(\xi)$ the completion of the fraction field of $\mathcal{A}/\text{Ker}(\xi)$. As in algebraic geometry if $Y := \mathcal{M}(\mathcal{A})$ the *functor of points of Y* associates to a Banach ring \mathcal{B} the set $Y(\mathcal{B})$ of bounded ring homomorphisms $\mathcal{A} \rightarrow \mathcal{B}$. The data of an individual point $\xi \in Y$ is equivalent to the data of the character $\chi_\xi : \mathcal{A} \rightarrow \mathcal{H}(\xi)$ since ξ can be recovered as the composite $\xi = |\cdot|_{\mathcal{H}(\xi)} \circ \chi_\xi$. Hence the restriction of the functor of points of Y to the category $E(K)$ is enough to distinguish the points of Y . In the sequel we only consider K -algebras \mathcal{A} topologically generated by K , by an element $T \in \mathcal{A}$, and by the elements of \mathcal{A} that are fraction of polynomials in $K[T] \subset \mathcal{A}$. Hence, if $\Omega \in E(K)$, a bounded K -linear character $\mathcal{A} \rightarrow \Omega$ will be determined by the image of T . In this situation the equivalence relation among characters indicated in [Ber90, 1.2.2,(ii)] is equivalent to say that two K -linear bounded characters $\chi : \mathcal{A} \rightarrow \Omega$ and $\chi' : \mathcal{A} \rightarrow \Omega'$ are equivalent if and only if there exists a larger extension $\Omega'' \in E(K)$, and two morphisms $\Omega \rightarrow \Omega''$ and $\Omega' \rightarrow \Omega''$ in the category $E(K)$, such that $\chi = \chi'$ as characters with values in Ω'' .

1.0.2 Disks. Let $\Omega \in E(K)$, $t \in \Omega$, $\rho \geq 0$. Consider the ring $\Omega\{\rho^{-1}(T - t)\}$ formed by power series $f(T) := \sum_{i \geq 0} a_i(T - t)^i$ such that $a_i \in \Omega$ for all $i \geq 0$, and $\lim_{i \rightarrow +\infty} |a_i|\rho^i = 0$. The setting $\xi_{t,\rho}(f) := \sup_{i \geq 0} |a_i|\rho^i$ is a multiplicative norm on $\Omega\{\rho^{-1}(T - t)\}$ that makes it a Banach Ω -algebra, and the Berkovich space $D^+(t, \rho) := \mathcal{M}(\Omega\{\rho^{-1}(T - t)\})$ is called the *closed disk*, centered at t with radius ρ . For all $\Omega' \in E(\Omega)$ the Ω' -valued points of $D^+(t, \rho)$ are given by $D_{\Omega'}^+(t, \rho) := \{t' \in \Omega' \text{ such that } |t' - t| \leq \rho\}$. The ring $\Omega\{\rho^{-1}(T - t)\}$ is a principal ideal domain, whose ideals are generated by polynomials in $\Omega[T]$. Moreover $\Omega[T] \subset \Omega\{\rho^{-1}(T - t)\}$ is dense. If t lies in K we say that the disk is K -rational, in this case $\Omega\{\rho^{-1}(T - t)\} = K\{\rho^{-1}(T - t)\} \hat{\otimes}_K \Omega$. The open disk $D^-(t, \rho)$ is the analytic space obtained by the union of $D^+(t, \rho')$ for all $\rho' < \rho$. Set $\mathcal{A}_\Omega(t, \rho) := \cap_{\rho' < \rho} \Omega\{(\rho')^{-1}(T - t)\}$. An element in this ring is a formal power series $\sum_{i \geq 0} a_i(T - t)^i$ with $a_i \in \Omega$ for all $i \geq 0$, satisfying $\lim_i |a_i|(\rho')^i = 0$ for all $\rho' < \rho$. The topology of $\mathcal{A}_\Omega(t, \rho)$ is defined by the family of norms $\{\xi_{t,\rho'}\}_{\rho' < \rho}$ (cf. [Bou98, Ch.IX, par. 1.2, Def.3, p.139]). If $\Omega' \in E(\Omega)$ one has $D_{\Omega'}^-(t, \rho) := \{t' \in \Omega' \text{ such that } |t' - t| < \rho\}$. As above $\Omega[T] \subset \mathcal{A}_\Omega(t, \rho)$ is dense.

1.0.3 Analytic elements, bounded analytic and analytic functions. Let $I \subset \mathbb{R}_{\geq 0}$ be an interval, $t \in \Omega$, and let $A(t, I) := \{|T - t| \in I\}$ be an annulus or disk. The ring $\mathcal{H}_\Omega(t, I)$ of (*Krasner*) *analytic elements* on $A(t, I)$ is the completion, under the sup-norm on $A(t, I)$, of rational fraction in $K(T)$ without poles in $A(t, I)$ (cf. [CR94], [Chr12], [Ked10b, Def.8.5.1]). The ring $\mathcal{A}_\Omega(t, I)$ of *analytic functions* on $A(t, I)$ is equal to $\mathcal{A}_\Omega(t, R)$ if $I = [0, R[$ (cf. section 1.0.2), while if $0 \notin I$, then it is formed by power series $\sum_{i \in \mathbb{Z}} a_i(T - t)^i$, with $a_i \in \Omega$, satisfying $\lim_{i \rightarrow \pm\infty} |a_i|\rho^i = 0$ for all $\rho \in I$ (cf. [CR94], [Chr12], [Ked10b, Def.8.4.2]). The ring $\mathcal{B}_\Omega(t, I)$ of *Bounded analytic functions* on $A(t, I)$ are analytic function satisfying $\sup_i |a_i|\rho^i \leq C < +\infty$ for all $\rho \in I$, where C is a convenient constant depending on the power series (cf. [CR94], [Chr12], [Ked10b, Def.8.1.5]).

1.1 Affinoid domains of the Berkovich affine line

$\mathbb{A}_K^{1,\text{an}}$ is the analytic space obtained as $\cup_{\rho \geq 0} D^+(0, \rho)$. Since $K[T]$ is dense on each $K\{\rho^{-1}T\}$, one proves that set theoretically $\mathbb{A}_K^{1,\text{an}}$ is constituted by the multiplicative semi-norms on $K[T]$ extending the absolute value of K (cf. [Ber90]). Such a multiplicative semi-norm is always given by

$$|f|_{t,\rho} := \xi_{t,\rho}(f) := \sup_{i \geq 0} \left| \frac{f^{(n)}(t)}{n!} \right|_\Omega \cdot \rho^i = \sup_{i \geq 0} \xi_t \left(\frac{f^{(n)}}{n!} \right) \cdot \rho^i, \quad (1.1)$$

for all $f \in K[T]$, where $\Omega \in E(K)$, $t \in \Omega$, and $\rho \geq 0$. Here ξ_t denotes $\xi_t(f) := |f(t)|_\Omega$ (cf. (1.6)). The choice of t and ρ is not unique, and in fact there is a canonical choice $\Omega := \mathcal{H}(\xi)$, t equal to

the image of T in $\mathcal{H}(\xi)$, and $\rho = 0$. In section 1.3.2 we give more details. An affinoid domain X of $\mathbb{A}_K^{1,\text{an}}$ is included in some closed disk $D^+(0, \rho)$, and is hence a Laurent domain of $D^+(0, \rho)$ (cf. [Ber90, 2.2.2]). If K is algebraically closed such a rational domain is of the form

$$X = D^+(c_0, R_0) - \cup_{i=1}^{\mu} D^-(c_i, R_i), \quad (1.2)$$

where $c_0, \dots, c_{\mu} \in K = \widehat{K^{\text{alg}}}$ satisfy $|c_i - c_0| \leq R_0$ for all $i = 1, \dots, \mu$, and $0 \leq R_1, \dots, R_{\mu} \leq R_0$. The ring $\mathcal{O}(X)$ of global section of X is the K -affinoid algebra $K\{\rho^{-1}T\}\{p^{-1}f, qg^{-1}\}$, where $p = R_0$, $f = T - c_0$, $q = (R_1, \dots, R_{\mu})$, $g = (T - c_1, \dots, T - c_{\mu})$ (cf. [Ber90, 2.2.2]). Denote by $\|\cdot\|_X$ its norm. By a Mittag-Leffler theorem [CR94, 5.3] the elements of $\mathcal{O}(X)$ can be uniquely write as $f = f_0 + f_1 + \dots + f_{\mu}$, where $f_0 = \sum_{i \geq 0} a_{0,i}(T - c_0)^i \in K\{\rho^{-1}(T - c_0)\}$, and, for all $j = 1, \dots, \mu$, $f_j = \sum_{i \geq 1} a_{j,i}(T - c_i)^{-i}$ with $a_{j,i} \in K$ and $\lim_{i \rightarrow \infty} |a_{j,i}|R_j^{-i} = 0$. The ring $\mathcal{O}^{\text{rat}}(X) := \mathcal{O}(X) \cap K(T)$ of rational fractions without poles in $X(K^{\text{alg}}) = \{t \in K^{\text{alg}} \text{ such that } |t - c_0| \leq R_0, \text{ and } |t' - c_i| \geq R_i\}$ is dense in $\mathcal{O}(X)$ with respect to $\|\cdot\|_X$.

LEMMA 1.1. *Let $\Omega \in E(K)$, $t \in \Omega$, $\rho \geq 0$. The semi-norm $\xi_{t,\rho} : K[T] \rightarrow \mathbb{R}_{\geq 0}$ extends by multiplicativity to $\mathcal{O}^{\text{rat}}(X)$, and by continuity to $\mathcal{O}(X)$ if and only if $t \in D_{\Omega}^+(c_0, R_0)$ and $\rho \in I_t$, where*

$$I_t := \begin{cases} [0, R_0] & \text{if } t \in X(\Omega) \\ [R_i, R_0] & \text{if } t \in D_{\Omega}^-(c_i, R_i) \end{cases} \quad \square \quad (1.3)$$

If $K \neq \widehat{K^{\text{alg}}}$, then $X \widehat{\otimes} \widehat{K^{\text{alg}}}$ is of the above form and $\mathcal{O}(X \widehat{\otimes} \widehat{K^{\text{alg}}}) = \mathcal{O}(X) \widehat{\otimes}_K \widehat{K^{\text{alg}}}$. By [Ber90, 1.3.6] one has $X \cong (X \widehat{\otimes} \widehat{K^{\text{alg}}})/G$, where $G := \text{Gal}(K^{\text{alg}}/K)$. Since $\text{Hom}_{\widehat{K^{\text{alg}}} - \text{Alg}}(\mathcal{O}(X) \widehat{\otimes} \widehat{K^{\text{alg}}}, \Omega) = \text{Hom}_{K - \text{Alg}}(\mathcal{O}(X), \Omega)$, for all $\Omega \in E(\widehat{K^{\text{alg}}})$ one still has

$$X(\Omega) = \{t \in \Omega \text{ such that } |t - c_0| \leq R_0, |t - c_i| \geq R_i\}, \quad (1.4)$$

for some $c_0, \dots, c_{\mu} \in K^{\text{alg}}$, $|c_i - c_0| \leq R_0$, and $0 < R_1, \dots, R_{\mu} \leq R_0$, as above, with the additional condition that $D_{\widehat{K^{\text{alg}}}}^+(c_0, R_0)$ is fixed by G , and the disks $\{D_{\widehat{K^{\text{alg}}}}^-(c_i, R_i)\}_{i=1, \dots, \mu}$ are permuted by G .

We say that X is *K-rational* if $c_0, \dots, c_n \in K$. One has $\mathcal{O}(X) = \mathcal{O}(X \widehat{\otimes} \widehat{K^{\text{alg}}})^G$, hence $\mathcal{O}(X)$ can be described as the completion of the ring $\mathcal{O}^{\text{rat}}(X) = (\mathcal{O}^{\text{rat}}(X \widehat{\otimes} \widehat{K^{\text{alg}}}))^G \subset K(T)$ of rational fractions without poles in $X(K^{\text{alg}})$ with respect to $\|\cdot\|_{X \widehat{\otimes} \widehat{K^{\text{alg}}}} = \max_{i=0, \dots, \mu} \xi_{c_i, R_i}$. Each $f \in \mathcal{O}(X)$ can be seen as a function on $X(\Omega)$ for all $\Omega \in E(K)$, and one has

$$\|f\|_X = \max_{i=0, \dots, \mu} \xi_{c_i, R_i}(f) = \sup_{\xi \in X} \xi(f) = \sup_{\Omega \in E(K), t \in X(\Omega)} |f(t)|_{\Omega}. \quad (1.5)$$

1.1.1 *Overconvergent functions on X.* For $\varepsilon > 0$ let $X_{\varepsilon} := D^+(c_0, R_0 + \varepsilon) - \cup_{i=1}^{\mu} D^-(c_i, R_i - \varepsilon)$ (over K or $\widehat{K^{\text{alg}}}$). We call overconvergent analytic functions on X the ring $\mathcal{O}^{\dagger}(X) := \cup_{\varepsilon > 0} \mathcal{O}(X_{\varepsilon})$. Differential modules over $\mathcal{O}^{\dagger}(X)$ are important especially because of the good properties of their de Rham cohomology. The material of this paper can be easily translate to the overconvergent case by replacing X with an X_{ε} with an unspecified choice of $\varepsilon > 0$ conveniently small.

1.2 Dwork generic points

For all $\Omega \in E(K)$ there is a natural map $i_{\Omega} : X(\Omega) \rightarrow X$ associating to $t \in X(\Omega)$ the semi-norm $f \mapsto |f(t)|_{\Omega}$. We will also use the following notation for the same semi-norm

$$i_{\Omega}(t)(f) = \xi_t(f) = \xi_{t,0}(f) = |f|_t = |f|_{t,0} = |f(t)|_{\Omega} = \lambda_t(0)(f) = |f(\xi_t)|. \quad (1.6)$$

If $j : \Omega' \rightarrow \Omega$ is a morphism in $E(K)$, then $i_{\Omega} \circ j = i_{\Omega'}$. For all semi-norm $\xi \in X$ there exists a $\Omega \in E(K)$ and a possibly not unique point $t \in X(\Omega)$ such that $\xi = |\cdot|_t$ i.e. $\xi(f) = |f(t)|_{\Omega}$ for all $f \in \mathcal{O}(X)$. Such a point t is called a *Dwork generic point* for ξ . As mentioned in the introduction there is a canonical choice given by $\Omega := \mathcal{H}(\xi)$, and t equal to the image t_{ξ} of T in $\mathcal{H}(\xi)$.

LEMMA 1.2. *Let $\xi \in X$. If $\Omega \in E(K)$ is algebraically closed and maximally complete, then $i_\Omega^{-1}(\xi)$ is an orbit under $\text{Gal}^{\text{cont}}(\Omega/K)$.*

Proof. Assume $\xi_t = \xi_{t'}$. Let $\widehat{K(t)}$ and $\widehat{K(t')}$ be the completions of the sub-fields of Ω generated by t and t' . Since the semi-norms $\xi_t = \xi_{t'}$ coincides on $K[T] \subset \mathcal{O}(X)$, then $\widehat{K(t)} \cong \mathcal{H}(\xi_t) = \mathcal{H}(\xi_{t'}) \cong \widehat{K(t')}$. Hence there exists a continuous isometric K -linear isomorphism $\sigma : \widehat{K(t)} \xrightarrow{\sim} \widehat{K(t')}$ such that $\sigma(t) = t'$. By [DR77, Lemma 8.3] σ extends to an isometric automorphism of Ω/K . \square

Recall that $\text{Gal}^{\text{cont}}(\Omega/K)$ acts isometrically on Ω . In the sequel of this paper Ω/K will be conveniently chosen, and often replaced by a larger one, without further specifications.

LEMMA 1.3. *Let $\Omega \in E(K)$, $t \in D_\Omega^+(c_0, R_0)$, $\rho \in I_t$ in order that $\xi_{t,\rho} \in X$ (cf. (1.3)). There exists a $\Omega' \in E(\Omega)$ and a Dwork generic point $t' \in X(\Omega')$ for $\xi_{t,\rho}$ satisfying $|t' - t| = \rho$.*

Proof. Let $\tilde{\xi}_{t,\rho} \in X \widehat{\otimes} \Omega$ be the lifting of $\xi_{t,\rho} \in X$ defined in the same way by (1.1). A Dwork generic point for $\tilde{\xi}_{t,\rho}$ is also a Dwork generic point for $\xi_{t,\rho}$. So we can assume $t \in K = \Omega$. In this case $T - t \in \mathcal{O}(X)$ hence $\rho = \xi_{t,\rho}(T - t) = \xi_{t'}(T - t) = |t' - t|$. \square

REMARK 1.4. *For $t \notin \widehat{K^{\text{alg}}}$, and $\rho < r(\xi_t)$ (cf. (1.9)), then any point t' satisfying $|t' - t| \leq \rho$ is a Dwork generic point of $\xi_{t,\rho} = \xi_{t,0}$.*

1.3 Canonical paths on X and generic radius of a point

Let $\Omega \in E(K)$ and $t \in D_\Omega^+(c_0, R_0)$. The path $\lambda_t : I_t \rightarrow X$ (cf. (1.1)) associating to ρ the Berkovich point $\lambda_t(\rho) := \xi_{t,\rho} = |\cdot|_{t,\rho}$ is continuous. More precisely let $f \in \mathcal{O}(X)$, then the map $\rho \mapsto |f|_{t,\rho} : I_t \rightarrow \mathbb{R}_{\geq 0}$ is continuous and enjoys the following properties:

- (LA) One has a partition $I_t = \cup_{k=1}^n I_k$, and $\alpha_1, \dots, \alpha_n \in |K|$, such that $|f|_{t,\rho} = \alpha_k \cdot \rho^{n_k}$, for all $\rho \in I_k$;
- (LC) Let $I \subseteq I_t$ be a subinterval. If the annulus $A(t, I) := \{|x - t| \in I\}$ is contained in $X \widehat{\otimes} \Omega$,⁷ then $n_{k+1} \geq n_k$ for all k such that $I_k \cap I \neq \emptyset$ and $I_{k+1} \cap I \neq \emptyset$;
- (Z) With the notation just introduced in (LC) assume that $I_k \cap I, I_{k+1} \cap I \neq \emptyset$, then f has exactly $n_{k+1} - n_k$ zeros $z \in K^{\text{alg}}$ such that $|z - t| = \sup I_k = \inf I_{k+1} \in |K^{\text{alg}}|$;⁸
- (M) Let $\Omega \in E(K)$, and let $a \in X(\Omega)$ be such that $|a - t|$ lies in the interior of I_k , and such that $D^-(a, |t - a|) \subset X$. Then one has $|f(a)|_\Omega = |f|_{a,|a-t|} = |f|_{t,|a-t|}$.

If $t \in X(\Omega)$, then $\xi_t \in X$ and $I_t = [0, R_0]$. Since $\xi_{t,\rho}$ is determined by continuity and multiplicativity by its restriction to $K[T]$, then by the last expression of (1.1) the path λ_t only depends on $\xi_t \in X$ and not on the choice of t . If $\xi = \xi_t$ from now on we indicate it by $\lambda_\xi := \lambda_t$.

The conditions (LA) and (LC) are known as log-affinity and log-convexity respectively. Namely in the sequel the log-function ${}^L h$ attached to a function $h : I_t \rightarrow \mathbb{R}_{\geq 0}$ will be by definition

$${}^L h := \ln \circ h \circ \exp : \ln(I_t) \rightarrow \mathbb{R} \cup \{-\infty\}, \quad (1.7)$$

where if $0 \in I_t = [0, R_0]$ (i.e. if $t \in X(\Omega)$), then by definition $\ln(I_t) = [-\infty, \ln(R_0)]$. We say that h has logarithmically a given property if ${}^L h$ has that property. Since $|f|_{t,\rho} \neq 0$ for all $\rho > 0$ it is often convenient to exclude the value $\rho = 0$, and consider ${}^L h$ as a function on $] - \infty, \ln(R_0)[$ with values in \mathbb{R} . If $t \in X(\Omega)$, and if $D^-(t, r) \subset X \widehat{\otimes} \Omega$, then $\rho \mapsto |f|_{t,\rho}$ is log-increasing for all $\rho \leq r$, and

$$|f|_{t,\rho} = \sup_{\Omega \in E(K), t' \in D_\Omega^-(t,\rho)} |f(t')|. \quad (1.8)$$

More precisely there exists a particular $\Omega \in E(K)$ such that $|f|_{t,\rho} = \sup_{t' \in D_\Omega^-(t,\rho)} |f(t')|$.

⁷Note that $A(t, I)$ is an Ω -rational analytic space. $A(t, I) \subset X \widehat{\otimes} \Omega$ if and only if there is no holes of X in $A(t, I)$.

⁸Note that the zeros of f are always algebraic, cf. [CR94].

1.3.1 *Generic radius of a point.* We call *generic radius* of ξ the number

$$r_K(\xi) := \max(\rho \in [0, R_0] \text{ such that } \lambda_\xi(\rho) = \lambda_\xi(0)). \quad (1.9)$$

We denote it by $r(\xi) := r_K(\xi)$ if no confusion is possible. The canonical path λ_ξ is constant on $[0, r(\xi)]$, and it induces an homeomorphism of $[r(\xi), R_0]$ with its image in X .

LEMMA 1.5. *Let $\xi \in X$, and let $t \in X(\Omega)$ be a Dwork generic point for t . Assume that $K^{\text{alg}} \subset \Omega$. Then $r(\xi)$ equals the distance of t from K^{alg} i.e. $r(\xi) = \inf_{c \in K^{\text{alg}}} |t - c|_\Omega$.*

Proof. Let $d_t := \inf_{c \in K^{\text{alg}}} |t - c|$. The zeros of any $f \in \mathcal{O}(X)$ are algebraic, then by the properties of section 1.3, $|f(t')| = |f(t)|$ for all $t' \in D_\Omega^-(t, d_t)$. So by (1.8) one has $\lambda_\xi(d_t) = \lambda_\xi(0)$ and hence $d_t \leq r(\xi)$. To show $r(\xi) \leq d_t$ observe that $\lambda_\xi(r(\xi)) = \lambda_\xi(0)$, so by (1.8) any polynomial in $K[T]$ has no zeros in $D_\Omega^-(t, r(\xi))$ (cf. (Z) of section 1.3), and so $D_\Omega^-(t, r(\xi)) \cap K^{\text{alg}}$ is empty. \square

If $\Omega' \in E(\Omega)$, each point in $D_{\Omega'}^-(t, r(\xi))$ is a Dwork generic point for ξ . For this reason $D^-(t, r(\xi))$ is called a *generic disk* for ξ (cf. section 1.3.3). We call X^{gen} the subset of X of points ξ satisfying $r(\xi) > 0$. A point ξ lies in $X - X^{\text{gen}}$ if and only if it admits a Dwork generic point in $X(\widehat{K^{\text{alg}}})$.

1.3.2 *Exact dependence of $\lambda_{\xi_t}(\rho)$ on the pair (t, ρ) .* Arguing as above one proves that if $t, t' \in D_\Omega^-(c_0, R_0)$ satisfy $\lambda_{\xi_t}(\rho) = \lambda_{\xi_{t'}}(\rho')$ for some $\rho \geq r(\xi_t)$, $\rho' \geq r(\xi_{t'})$, then $\rho = \rho'$, and $\lambda_{\xi_t}(r) = \lambda_{\xi_{t'}}(r)$ for all $r \geq \rho$. Moreover, up to enlarge Ω , there exists an automorphism $\sigma \in \text{Gal}^{\text{cont}}(\Omega/K)$ such that $|\sigma(t') - t| \leq \rho$. Reciprocally if for some $\sigma \in \text{Gal}^{\text{cont}}(\Omega/K)$ one has $|\sigma(t') - t| \leq \rho$, then $\lambda_t(r) = \lambda_{t'}(r)$ for all $r \geq \rho$. With this description one proves the following

LEMMA 1.6. *One has $r(\xi_{t,\rho}) = \max(\rho, r(\xi_t))$, in particular if $t \in X(\widehat{K^{\text{alg}}})$, then $r(\xi_{t,\rho}) = \rho$.* \square

1.3.3 *Image in the Berkovich space of an open disk or annulus. Tangent disks.* Let $D^-(t, \rho)$ be an Ω -rational open disk contained in $X \widehat{\otimes} \Omega$ (this amounts to ask $\rho \leq \rho_{t,X}$, cf. section 1.5). If $\rho \leq r(\xi_t)$ the image of $D^-(t, \rho)$ in X by the canonical map $X \widehat{\otimes} \Omega \rightarrow X$ is reduced to $\{\xi_t\}$. In this case we say that the open disk is *generic*. On the other hand if $\rho > r(\xi_t)$, then by Lemma 1.5, up to enlarge Ω , there is a center of the disk in K^{alg} . So the image of $D^-(t, \rho)$ in X is a genuine K^{alg} -rational disk. In both cases the image of $D^-(t, \rho)$ in X equals $\cup_{\Omega' \in E(\Omega)} i_{\Omega'}(D_{\Omega'}^-(t, \rho))$ (cf. (1.6)).

Let $S \subseteq X$ be a subset. We say that an open disk $D^-(t, \rho) \subset X \widehat{\otimes} \Omega$ is *tangent to S at $\xi \in S$* if $\xi = \xi_{t,\rho}$ and if either the disk is generic with $\rho = r(\xi)$ or, if it is not generic, the intersection of S with the image of $D^-(t, \rho)$ in X is empty. By section 1.3.2 all open disks tangent to ξ have the same radius $\rho = r(\xi)$.

1.3.4 *Description of $X \widehat{\otimes} \widehat{K^{\text{alg}}} \rightarrow X$.* Sections 1.3.3 and 1.3.2 provide a complete description of the map $X \widehat{\otimes} \widehat{K^{\text{alg}}} \rightarrow X$. Indeed if $t \in X(K^{\text{alg}})$ the Galois group $G := \text{Gal}(K^{\text{alg}}/K)$ acts on the branch $\Lambda(\xi_t)$ as $\sigma(\lambda_{\xi_t}(\rho)) := \lambda_{\xi_t}(\rho) \circ \sigma = \lambda_{\xi_{\sigma(t)}}(\rho)$. On the other hand every semi-norm $\xi \in X \widehat{\otimes} \widehat{K^{\text{alg}}}$ is the infimum of a totally ordered family of semi-norms $\lambda_t(\rho)$ with $t \in X(K^{\text{alg}})$ (cf. [Ber90, 1.4.4]), and since G preserves the diameters ρ , then it commutes with the infimum.

1.4 Branches and saturated subsets.

A *closed branch* $\Lambda(\xi)$ with starting point $\xi \in X$ is the image in X of the interval $[0, R_0]$ by the canonical path λ_ξ . A *branch* is the union of a totally ordered family of closed branches by inclusions of subsets. The union of such branches is the whole space X (cf. section 1.2). An *open or closed segment* of a branch $\Lambda(\xi)$ is the image $\lambda_\xi(I) \subset X$ of a open or closed interval I of \mathbb{R} which is contained in $[r(\xi), R_0]$. A *saturated* subset Γ of X is by definition an arbitrary union of branches

of X . The family of all branches of X contained in Γ is a partially ordered set by the inclusion of subsets of X . A *maximal branch* of Γ is a maximal element of this family. Each point of Γ is contained in a maximal branch of Γ which is hence the union of its maximal branches. We say that Γ is *branch-closed* if its maximal branches are all closed. Finally Γ is called *finite* if it has a finite number of maximal branches. Each branch $\Lambda(\xi)$ always intersects all saturated subsets of X because all branches have a common point $\lambda_\xi(R_0) = |\cdot|_{c_0, R_0}$. If $S \subseteq X$ is a subset, we denote by $\text{Sat}(S) = \cup_{\xi \in S} \Lambda(\xi)$ the smallest saturated subset containing S . We say that Γ is *K-rational* if $\Gamma = \text{Sat}(S)$ with $S = \{\xi_{t_i, \rho_i}\}_{i \in I}$ and for all $i \in I$ one has $t_i \in X(K)$ (I is possibly not finite).

DEFINITION 1.7. Let Γ be a saturated subset and let $|\cdot| \in X$. We denote by

$$\rho_\Gamma(\xi) := \inf(\rho \geq r(\xi) \text{ such that } \lambda_\xi(\rho) \in \Gamma) . \quad (1.10)$$

Define $\delta_\Gamma(\xi) := \lambda_\xi(\rho_\Gamma(\xi))$. The map $\delta_\Gamma : X \rightarrow X$ is the identity on the smallest branch-closed saturated subset $\bar{\Gamma}$ containing Γ , and $\delta_\Gamma(X) = \bar{\Gamma}$. If $\Gamma = \bar{\Gamma}$ we call $\delta_\Gamma : X \rightarrow \Gamma$ the *canonical retraction*. If $\Gamma = \bar{\Gamma}$ is non empty, branch-closed, and finite, and if it is equipped with the quotient topology induced by the topology of $[0, R_0] \subset \mathbb{R}$ via the canonical paths λ_ξ , then δ_Γ is continuous, and moreover Γ is the topological quotient of X by the map $\delta_\Gamma : X \rightarrow \Gamma$.

1.5 The skeleton of X and the function $\rho_{-,X}$.

The *skeleton* $\Gamma_X = \text{Sat}(S)$ of X is the smallest finite branch-closed saturated subset containing the Shilov boundary $S = \{\xi_{c_i, R_i}\}_{i=0, \dots, \mu}$ (cf. (1.2)). Γ_X is also the set of semi-norms of X that are maximal with respect to the partial order given by $\xi \leq \xi'$ if and only if $\xi(f) \leq \xi'(f)$ for all $f \in \mathcal{O}(X)$. We denote by $\rho_{\xi, X} := \rho_{\Gamma_X}(\xi)$ (cf. Def. 1.7). If $t \in X(\Omega)$ is a Dwork generic point for ξ one has

$$\rho_{\xi, X} = \rho_{t, X} := \min_{i=1, \dots, \mu} (|t - c_i|_\Omega, R_0) . \quad (1.11)$$

This expression represents $\rho_{\xi, X}$ as the radius ρ of the largest open disk $D^-(t, \rho)$ contained in $X \hat{\otimes} \Omega$. If $\xi \leq \xi'$, then $\rho_{\xi, X} = \rho_{\xi', X}$. In fact the inequality $\xi \leq \xi'$ applied to $T - c_i$ and $(T - c_i)^{-1}$ provides $|t - c_i| = |t' - c_i|$ in (1.11). For all $t \in X(\Omega)$, and all $\rho \geq 0$ one has $\rho_{\xi_{t, \rho}, X} = \max(\rho, \rho_{\xi_t, X})$.

REMARK 1.8. For a closed disk this definition differs from [Ber90, 1.4] which gives $\Gamma_X = \emptyset$.

1.6 Directions and directional finiteness.

Let $X_{\text{int}} \subset X^{\text{gen}} \subset X$ be the set of Berkovich points ξ of the form $\xi = \lambda_{\xi'}(\rho)$, with $\rho > r(\xi')$. In other words $X_{\text{int}} \subset X$ is formed by the points $\xi_{t, \rho}$ defined by a *non generic* open disk $D^-(t, \rho) \subset X$. These are the points of type (2) and (3) in the terminology of [Ber90, 1.4.4]. For $\xi \in X_{\text{int}}$ we denote by $\mathcal{B}(\xi)$ be the family of branches $\Lambda(\xi')$ admitting an open segment containing ξ i.e. such that $\xi = \lambda_{\xi'}(\bar{\rho})$ for some $\bar{\rho} > r(\xi')$. One defines an equivalence relation on $\mathcal{B}(\xi)$ as follows. We say that $\Lambda(\xi_1) \sim \Lambda(\xi_2)$ if and only if the two paths meet before meeting ξ i.e. if and only if there exists $\rho < \bar{\rho}$ such that $\lambda_{\xi_1}(\rho) = \lambda_{\xi_2}(\rho)$. A *direction through* ξ is an equivalence class in $\Delta(\xi) := \mathcal{B}(\xi) / \sim$. A *representative branch* $\Lambda(\xi_\delta)$ for a direction $\delta \in \Delta(\xi)$ is an arbitrary element of the class δ . Let $\Gamma \subseteq X$ be a saturated subset, and let $\xi \in \Gamma \cap X_{\text{int}}$. We say that a *direction* $\delta \in \Delta(\xi)$ *belongs to* Γ if there exists a representative branch $\Lambda(\xi_\delta)$ for δ contained in Γ . The set of directions belonging to Γ will be denoted by $\Delta(\xi, \Gamma)$. If $\Delta(\xi, \Gamma)$ is finite, we say that Γ is *directionally finite* at ξ .

REMARK 1.9. The extra direction directed toward the infinity is not considered here as a direction.

REMARK 1.10. Since $\bar{\rho} > r(\xi')$, then, by section 1.3.2, one can chose $\xi' = \xi_t$ with $t \in X(K^{\text{alg}})$. Hence if $\xi \in X_{\text{int}}$ and if there exists two distinct direction $\delta, \delta' \in \Delta(\xi)$, then $\xi = \xi_{t, \bar{\rho}} = \xi_{t', \bar{\rho}}$ with $t, t' \in X(K^{\text{alg}})$ and $\bar{\rho} = |t - t'| \in |K^{\text{alg}}| - \{0\}$. ξ is then a point of type (2) in (cf. [Ber90, 1.4.4]). If

$K = \widehat{K^{\text{alg}}}$, then by a translation and a dilatation of $\mathbb{A}_K^{1,\text{an}}$, ξ is sent in $\xi_{0,1}$, and $\Delta(\xi)$ is identified to $\Delta(\xi_{0,1})$. This will be the case of bifurcation points of the skeleton of a function (cf. section 2.2).

2. Constancy skeleton of a function on X .

Let \mathcal{T} be a set and let $\mathcal{R} : X \rightarrow \mathcal{T}$ be an arbitrary function. For all $\xi \in X$ consider the composite map $\mathcal{R}_\xi : X \widehat{\otimes} \mathcal{H}(\xi) \rightarrow X \rightarrow \mathcal{T}$, and define the *constancy radius* $\rho_{\mathcal{R}}(\xi)$ of \mathcal{R} at ξ as the maximum value of ρ such that \mathcal{R}_ξ is constant on the open disk $D^-(t_\xi, \rho) \subset X \widehat{\otimes} \mathcal{H}(\xi)$, where t_ξ is the image of T in $\mathcal{H}(\xi)$. Define the *constancy skeleton*, or simply the *skeleton*, $\Gamma(X, \mathcal{R}) \subseteq X$ of \mathcal{R} as the set of points of X of the form $\lambda_\xi(\rho_{\mathcal{R}}(\xi)) \in X$. We write $\Gamma(\mathcal{R})$ if no confusion is possible. Since $D^-(t_\xi, r(\xi))$ is contained in the inverse image of ξ , then from the definition one immediately has

$$r(\xi) \leq \rho_{\mathcal{R}}(\xi) \leq \rho_{\xi, X} \leq R_0. \quad (2.1)$$

2.0.1 Functorial point of view. By composing with i_Ω (cf. (1.6)) we obtain for all $\Omega \in E(K)$ a map $\mathcal{R}_\Omega : X(\Omega) \rightarrow X \rightarrow \mathcal{T}$ which is obviously compatible with the inclusions of CVFE's of K . In other words the family $\{\mathcal{R}_\Omega\}_\Omega$ is a natural transformation between the functor $X : \Omega \mapsto X(\Omega)$ and the constant functor $\Omega \mapsto \mathcal{T}$. Let $\Omega \in E(K)$, $t \in X(\Omega)$ and $\rho > 0$. We say that \mathcal{R} is *constant on the open disk* $D^-(t, \rho) \subset X \widehat{\otimes} \Omega$, if for all $\Omega' \in E(\Omega)$ the function $\mathcal{R}_{\Omega'} : D_{\Omega'}^-(t, \rho) \rightarrow \mathcal{T}$ is constant. We define the *constancy radius* $\rho_{\mathcal{R}}(t)$ of \mathcal{R} at t as the radius of the largest open disk $D^-(t, \rho_{\mathcal{R}}(t))$ contained in $X \widehat{\otimes} \Omega$ (i.e. $\rho_{\mathcal{R}}(t) \leq \rho_{t, X}$) on which \mathcal{R} is constant. If $\rho_{\mathcal{R}, \Omega'}(t) \leq \rho_{\xi_t, X}$ denotes the largest radius such that $\mathcal{R}_{\Omega'}$ is constant on the set $D_{\Omega'}^-(t, \rho_{\mathcal{R}, \Omega'}(t))$, then

$$\rho_{\mathcal{R}}(t) := \inf_{\Omega' \in E(\Omega)} \rho_{\mathcal{R}, \Omega'}(t). \quad (2.2)$$

LEMMA 2.1. *This definition coincides with that of the above section : $\rho_{\mathcal{R}}(t) = \rho_{\mathcal{R}}(\xi_t)$.*

Proof. Firstly we prove the independence on t . Let t, t' be two Dwork generic points for ξ . Up to enlarge Ω' one can assume that $t, t' \in X(\Omega')$ and that there exists $\sigma \in \text{Gal}^{\text{cont}}(\Omega'/K)$ such that $t' = \sigma(t)$. Since σ is isometric one has $\sigma(D_{\Omega'}^-(t, \rho)) = D_{\Omega'}^-(\sigma(t), \rho)$. By construction $\mathcal{R}_{\Omega'}$ is constant on the orbit $i_{\Omega'}^{-1}(\xi')$, for all $\xi' \in X$. So $\mathcal{R}_{\Omega'}$ is constant on $D_{\Omega'}^-(t, \rho)$ if and only if it is constant on $\sigma(D_{\Omega'}^-(t, \rho))$. This proves that $\rho_{\mathcal{R}, \Omega'}(t) = \rho_{\mathcal{R}, \Omega'}(t')$, and since this holds for all $\Omega' \in E(\Omega)$ large enough one also has $\rho_{\mathcal{R}}(t) = \rho_{\mathcal{R}}(t')$ by (2.2). Equality $\rho_{\mathcal{R}}(t) = \rho_{\mathcal{R}}(\xi_t)$ then follows from the fact that the image of a disk $D^-(t, \rho)$ in X is equal to $\cup_{\Omega' \in E(\Omega)} i_{\Omega'}(D_{\Omega'}^-(t, \rho))$ (cf. section 1.3.3). \square

2.0.2 Basic properties.

PROPOSITION 2.2. $\Gamma(\mathcal{R})$ is a saturated subset satisfying moreover:

- i) $\Gamma(\mathcal{R})$ is branch-closed (i.e. its maximal branches are closed);
- ii) $\Gamma_X \subseteq \Gamma(\mathcal{R})$ (i.e. $\Gamma(\mathcal{R})$ always contains the skeleton of X , cf. section 1.5);
- iii) $\xi \in \Gamma(\mathcal{R})$ if and only if $\rho_{\mathcal{R}}(\xi) = r(\xi)$;
- iv) $\lambda_\xi(\rho_{\mathcal{R}}(\xi)) \in \Gamma_X$ if and only if $\rho_{\mathcal{R}}(\xi) = \rho_{\xi, X}$;
- v) $\rho_{\mathcal{R}}(\xi) = \rho_{\Gamma(\mathcal{R})}(\xi)$ for all $\xi \in X$ (cf. (1.10));
- vi) For all $\xi \in X$, and all $\rho \in [0, R_0]$ one has $\rho_{\mathcal{R}}(\lambda_\xi(\rho)) = \max(\rho, \rho_{\mathcal{R}}(\xi))$.
- vii) If \mathcal{R} is constant on a non generic disk $D^-(t, \rho) \subset X$, $t \in X(\Omega)$, then $D^-(t, \rho) \cap \Gamma(\mathcal{R})$ is empty.

Proof. One can assume $K = \widehat{K^{\text{alg}}}$. Property vii) is evident. This implies iii) because if $r(\xi) < \rho_{\mathcal{R}}(\xi)$ then $\xi \notin \Gamma(\mathcal{R})$, since $\xi \in D^-(t_\xi, \rho_{\mathcal{R}}(\xi))$. Conversely if $r(\xi) = \rho_{\mathcal{R}}(\xi)$, then $\xi = \lambda_\xi(r(\xi)) = \lambda_\xi(\rho_{\mathcal{R}}(\xi)) \in \Gamma(\mathcal{R})$. By (2.1) iii) implies ii) since $\xi \in \Gamma_X$ if and only if $r(\xi) = \rho_{\xi, X}$. These properties imply that

$\Gamma(\mathcal{R})$ is saturated. In fact if $\xi \in \Gamma_X$, then $\Lambda(\xi) \subseteq \Gamma_X \subseteq \Gamma(\mathcal{R})$ because Γ_X is saturated. On the other hand let $\xi \in \Gamma(\mathcal{R}) - \Gamma_X$, and let $\xi' := \lambda_\xi(\rho) \notin \Gamma_X$. Then one must have $\rho_{\mathcal{R}}(\xi') = r(\xi')$ and hence $\xi' \in \Gamma(\mathcal{R})$, because otherwise \mathcal{R} would be constant on the non generic disk $D^-(t_{\xi'}, \rho_{\mathcal{R}}(\xi'))$ which contains $\xi \in \Gamma(\mathcal{R})$, contradicting vii). To prove i) let Λ be a maximal branch. If $\Lambda \subseteq \Gamma_X$ there is nothing to prove since $\Lambda = \Lambda(\xi_{c_i, R_i})$. If $\Lambda \not\subseteq \Gamma_X$, then we can express Λ as the disjoint union of a closed segment J with a segment I such that $I \cap \Gamma_X$ is empty. We claim that the infimum semi-norm $\xi := \inf_{\xi' \in I} \xi'$ belongs to $\Gamma(\mathcal{R})$ and hence to I . In fact $\inf_{\xi' \in I} \xi' = \inf_{r(\xi) < \rho < \rho_{\xi, X}} \lambda_\xi(\rho)$, and by (2.1) $r(\xi) \leq \rho_{\mathcal{R}}(\xi)$. If by contrapositive $r(\xi) < \rho_{\mathcal{R}}(\xi)$, then \mathcal{R} is constant on a non generic disk $D^-(t_\xi, \rho)$ with $\rho > r(\xi)$ close enough to $r(\xi)$, and $\xi' := \lambda_\xi(\rho)$ do not belongs to $\Gamma(\mathcal{R})$ by vii). So $r(\xi) = \rho_{\mathcal{R}}(\xi)$ and $\xi \in \Gamma(\mathcal{R})$ by iii). v) and vi) are straightforward. \square

We denote by

$$\delta_{\mathcal{R}} : X \rightarrow \Gamma(\mathcal{R}) \quad (2.3)$$

the canonical retraction defined by $\delta_{\mathcal{R}}(\xi) := \lambda_\xi(\rho_{\mathcal{R}}(\xi)) = \delta_{\Gamma(\mathcal{R})}(\xi)$. The map $\delta_{\mathcal{R}}$ is well defined since $\Gamma(\mathcal{R})$ is branch-closed. We say that \mathcal{R} is *finite* if $\Gamma(\mathcal{R})$ is a finite saturated subset. If \mathcal{R} is finite one proves easily that $\delta_{\mathcal{R}}$ is a continuous map.

REMARK 2.3. *The correspondence $\mathcal{R} \mapsto \delta_{\mathcal{R}}$ is idempotent (i.e. $\delta_{\delta_{\mathcal{R}}} = \delta_{\mathcal{R}}$). More precisely if $\Gamma \subseteq X$ is a saturated subset, and if $\mathcal{R} = \delta_\Gamma : X \rightarrow \bar{\Gamma}$ is its retraction, then $\delta_{\delta_\Gamma} = \delta_{\Gamma \cup \Gamma_X}$. Any branch-closed saturated subset Γ of X containing Γ_X is the skeleton of its retraction map δ_Γ (i.e. $\Gamma = \Gamma(\delta_\Gamma)$).*

REMARK 2.4. *Let $\mathcal{R}_i : X \rightarrow \mathcal{T}_i$, $i = 1, 2$, and let $g : \mathcal{T}_1 \times \mathcal{T}_2 \rightarrow \mathcal{T}_3$ be any functions. If $\mathcal{R}_3 := g \circ (\mathcal{R}_1 \times \mathcal{R}_2)$, then $\Gamma(\mathcal{R}_3) \subseteq \Gamma(\mathcal{R}_1) \cup \Gamma(\mathcal{R}_2)$. Indeed clearly $\rho_{\mathcal{R}_3}(\xi) \geq \min(\rho_{\mathcal{R}_1}(\xi), \rho_{\mathcal{R}_2}(\xi))$, and $\Gamma(\mathcal{R}_1) \cup \Gamma(\mathcal{R}_2)$ is saturated. This holds in particular for $\max(\mathcal{R}_1, \mathcal{R}_2)$ or $\min(\mathcal{R}_1, \mathcal{R}_2)$ if $\mathcal{T}_i = \mathbb{R}$.*

REMARK 2.5. *Let $X' \subseteq X$ a sub-affinoid, and $\mathcal{R}' : X' \rightarrow \mathcal{T}$ be the restriction of $\mathcal{R} : X \rightarrow \mathcal{T}$ to X' . To avoid confusion we denote by $\Gamma(X, \mathcal{R}) \subseteq X$, $\Gamma(X', \mathcal{R}') \subseteq X'$, $\rho_{\mathcal{R}}(X, -)$, $\rho_{\mathcal{R}'}(X', -)$ the respective skeletons and constancy radii. For all $\xi' \in X'$ one clearly has $\rho_{\mathcal{R}'}(X', \xi') = \min(\rho_{\mathcal{R}}(X, \xi'), \rho_{\xi', X'})$, hence $\Gamma(X', \mathcal{R}') = (\Gamma(X, \mathcal{R}) \cap X') \cup \Gamma_{X'}$. So the finiteness of \mathcal{R} on X implies that of \mathcal{R}' on X' .*

2.0.3 Examples of skeletons.

- i) Let $\mathcal{R} = \text{Id}_X : X \rightarrow X$ be the identity, then $\Gamma(\text{Id}_X) = \Gamma(r(-)) = X$ (cf. (1.9)).
- ii) Let $\mathcal{R} = 1 : X \rightarrow \{pt\}$ be a constant map, then $\Gamma(1) = \Gamma(\rho_{-, X}) = \Gamma_X$ is the skeleton of X .
- iii) Let $f_1, \dots, f_n \in \mathcal{O}(X)$, let $\alpha_1, \dots, \alpha_n > 0$, and let $\mathcal{R}(\xi) := \min_i (|f_i(\xi)|^{\alpha_i})$. Then $\Gamma(\mathcal{R}) = \text{Sat}(\{z_1, \dots, z_r\}) \cup \Gamma_X$, where $\{z_1, \dots, z_r\} \subset X(K^{\text{alg}})$ is the union of all zeros of f_1, \dots, f_n .
- iv) With the above notations if $\mathcal{R}(\xi) := \max_i (|f_i(\xi)|^{-\alpha_i})$, intended as a function with values in the set $\mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\}$, then one again has $\Gamma(\mathcal{R}) = \text{Sat}(\{z_1, \dots, z_r\}) \cup \Gamma_X$.
- v) Assume now that $\mathcal{R}(\xi) := \max_i |f_i(\xi)|^{\alpha_i}$ (resp. $\mathcal{R}(\xi) := \min_i |f_i(\xi)|^{-\alpha_i}$) as a function with values in $\mathcal{T} := \mathbb{R}_{>0} \cup \{\infty\}$. In this case the explicit description of the skeleton $\Gamma(\mathcal{R})$ is more complicate, but one can easily deduce its finiteness from Remark 2.4.

2.1 Branch continuity and dag-skeleton.

We investigate now whether the function \mathcal{R} admits a factorization as $\mathcal{R} = \mathcal{R}_{|\Gamma(\mathcal{R})} \circ \delta_{\mathcal{R}}$:

$$\begin{array}{ccc} & X & \xrightarrow{\mathcal{R}} \mathcal{T} \\ & \searrow \delta_{\mathcal{R}} & \\ \Gamma(\mathcal{R}) & \xlongequal{\quad} \Gamma(\mathcal{R}) & \xrightarrow{\mathcal{R}_{|\Gamma(\mathcal{R})}} \mathcal{T} \end{array} \quad (2.4)$$

This is not automatically verified. In fact for a given $\xi \in X$ the restriction $\mathcal{R} \circ \lambda_\xi : [0, R_0] \rightarrow \mathcal{T}$ is constant for $\rho \in [0, \rho_{\mathcal{R}}(\xi)[$, but one may have a different value at $\rho = \rho_{\mathcal{R}}(\xi)$. We say that \mathcal{R} is *branch continuous* if for all $\xi \in X$ one has $\mathcal{R}(\lambda_\xi(\rho_{\mathcal{R}}(\xi))) = \lim_{\rho \rightarrow \rho_{\mathcal{R}}(\xi)^-} \mathcal{R}(\lambda_\xi(\rho)) = \mathcal{R}(\xi)$. A branch continuous map factorizes as $\mathcal{R} = \mathcal{R}_{|\Gamma(\mathcal{R})} \circ \delta_{\mathcal{R}}$ and is determined by its values on $\Gamma(\mathcal{R})$. A continuous function with values in a Hausdorff space \mathcal{T} is branch continuous. Conversely a finite and branch continuous function is continuous if and only if its restriction to $\Gamma(\mathcal{R})$ is continuous.

For some purposes this situation may be unsatisfactory since we want to factorize all functions. For this we define the *dag-skeleton* $\Gamma(\mathcal{R})^\dagger$ as follows. A *germ of a direction* of $\delta \in \Delta(\xi)$ is an arbitrary *unspecified* representative branch $\Lambda(\xi_\delta)$ for δ (cf. section 1.6). We define $\Gamma(\mathcal{R})^\dagger$ as the union of the skeleton $\Gamma(\mathcal{R})$ together with a *germ of each directions* $\delta \in \Delta(\xi)$ of each point ξ of $\Gamma(\mathcal{R})$. Any function \mathcal{R} factorizes through its dag-skeleton $\Gamma(\mathcal{R})^\dagger$. This situation will not occur in this paper since all the functions will be branch continuous. This idea can be better expressed in term of Huber spaces [Hub96], but this lies outside the scopes of this paper.

2.2 Boundary, bifurcation points, and smooth points.

We distinguish three kind of points in $\Gamma(\mathcal{R})$:

- i) The *boundary* of $\Gamma(\mathcal{R})$ is by definition constituted by ξ_{c_0, R_0} , and by those points $\xi \in X$ such that $\Lambda(\xi)$ is a maximal branch of $\Gamma(\mathcal{R})$.
- ii) A *bifurcation point* of $\Gamma(\mathcal{R})$ is by definition a point $\xi \in \Gamma(\mathcal{R}) \cap X_{\text{int}}$ for which there exists at least two distinct directions $\delta_1, \delta_2 \in \Delta(\xi)$ belonging to $\Gamma(\mathcal{R})$ (cf. section 1.6). The unique point which is possibly simultaneously a bifurcation and boundary point of $\Gamma(\mathcal{R})$ is ξ_{c_0, R_0} .
- iii) We call *smooth* point of $\Gamma(\mathcal{R})$ any other point of $\Gamma(\mathcal{R})$.
- iv) Among smooth point there are those called *punctured smooth*. These are the smooth points ξ for which there exists at least a hole $D^-(c_i, R_i)$ of X which is tangent to $\Gamma(\mathcal{R})$ at $\xi = \xi_{c_i, R_i}$ (cf. section 1.3.3). Punctured smooth points of $\Gamma(\mathcal{R})$ are the smooth points of $\Gamma(\mathcal{R})$ belonging to the Shilov boundary on Γ_X .

REMARK 2.6. Functions $f \in \mathcal{O}(X)$ may not be log-convex along $\Gamma(\mathcal{R})$ in correspondence of a punctured smooth point (cf. property (LC) of section 1.3). Moreover if one works with over-convergent functions $\mathcal{O}^\dagger(X)$ on X , then punctured smooth points have to be considered as bifurcation points.

REMARK 2.7. Bifurcation points are all of type (2) in the sense of [Ber90, 1.4.4] (cf. Remark 1.10).

DEFINITION 2.8. We call *critical point* of X , denoted by \mathcal{C}_X the points of the Shilov boundary of X together with the bifurcation points of Γ_X . Explicitly one has $\mathcal{C}_X := \{\xi_{c_i, R_i}\}_{i=0, \dots, \mu} \cup \{\xi_{c_i, |c_i - c_j|}\}_{i \neq j, i, j=1, \dots, \mu}$ (cf. (1.2)). More generally if Γ is any branch-closed saturated subset containing Γ_X we call $\mathcal{C}(\Gamma)$ the union of \mathcal{C}_X with the set of bifurcation and boundary points of Γ .

2.3 A Criterion for the finiteness of a real valued function \mathcal{R} .

Let $\mathcal{R} : X \rightarrow \mathbb{R}_{>0}$ be any function. For all path λ_ξ we indicate by

$$\mathcal{R}_\xi := \mathcal{R} \circ \lambda_\xi : [0, R_0] \rightarrow \mathbb{R}_{>0} \quad (2.5)$$

the composite map. If t is a Dwork generic point for ξ we also use the notation $\mathcal{R}_t := \mathcal{R}_\xi$. By definition \mathcal{R}_ξ is constant on $[0, r(\xi)]$. The log-function ${}^L\mathcal{R}_\xi :]-\infty, \ln(R_0)] \rightarrow \mathbb{R}$ attached to \mathcal{R}_ξ is given by (cf. (1.7))

$${}^L\mathcal{R}_\xi(\tau) := \ln(\mathcal{R}_\xi(\exp(\tau))) , \quad \tau \in]-\infty, \ln(R_0)] \quad (2.6)$$

Let $\xi \in X_{\text{int}}$, and let $\Lambda(\xi_\delta) \in \mathcal{B}(\xi)$ be a representative branch for a direction $\delta \in \Delta(\xi)$ through ξ . The left slope of ${}^L\mathcal{R}_{\xi_\delta}$ at ξ (if it exists) only depends on the direction δ defined by $\Lambda(\xi_\delta)$. Let now

$\xi \in X - \{\xi_{c_0, R_0}\}$, the right slope of ${}^L\mathcal{R}_\xi$ at ξ (if it exists) only depend on the branch $\Lambda(\xi)$. We denote them by

$$\partial_+\mathcal{R}(\xi) := \lim_{\tau \rightarrow \tau_0^+} \frac{{}^L\mathcal{R}_\xi(\tau) - {}^L\mathcal{R}_\xi(\tau_0)}{\tau - \tau_0}, \quad \partial_-\mathcal{R}_\delta(\xi) := \lim_{\tau \rightarrow \bar{\tau}^-} \frac{{}^L\mathcal{R}_{\xi_\delta}(\tau) - {}^L\mathcal{R}_{\xi_\delta}(\bar{\tau})}{\tau - \bar{\tau}} \quad (2.7)$$

where $\bar{\tau} \in]-\infty, \ln(R_0)]$ is defined by the relation $\xi = \lambda_{\xi_\delta}(\exp(\bar{\tau}))$, and $\tau_0 := \ln(r(\xi))$

DEFINITION 2.9 (Flat directions). *We say that a direction $\delta \in \Delta(\xi)$ is flat for \mathcal{R} if $\partial_-\mathcal{R}_\delta(\xi) = 0$. If all directions are flat for \mathcal{R} , and if $\partial_+\mathcal{R}(\xi) = 0$ too, we say that \mathcal{R} is flat at ξ .*

2.3.1 *Conditions and criterion.* Let as usual $X = D^+(c_0, R_0) - \cup_{i=1}^\mu D^-(c_i, R_i)$. Let $\mathcal{R} : X \rightarrow \mathbb{R}_{>0}$ be a function. Let Γ be a finite and branch-closed saturated subset containing Γ_X . Consider the following conditions:

- (C1) For all $\xi \in X$ one has $\rho_{\mathcal{R}}(\xi) > 0$. We say that \mathcal{R} is *locally constant*. By (2.1) this condition is automatically verified by all $\xi \in X^{\text{gen}}$.
- (C2) For all $\xi \in X$ the function ${}^L\mathcal{R}_\xi :]-\infty, \ln(R_0)] \rightarrow \mathbb{R}_{>0}$ is piecewise linear and continuous, with a finite number of breaks. Note that if \mathcal{R} verifies (C1) and (C2) then it is branch-continuous.
- (C3) Let $D^-(t, \rho) \subset X$ be a non generic disk such that $D^-(t, \rho) \cap \Gamma = \emptyset$. Then ${}^L\mathcal{R}_t$ is concave on $]0, \ln(\rho)[$. Note that concavity implies continuity on $]0, \rho[$, hence the left and right slopes of ${}^L\mathcal{R}_t$ along $] -\infty, \ln(\rho)[$ exists and are finite.
- (C4) The modulus of all possible *non zero* slopes of \mathcal{R} at any point is lower bounded by a positive real number $\nu_{\mathcal{R}} > 0$, which is independent on the Berkovich point. Namely for all $\xi \in X_{\text{int}}$ (resp. $\xi \in X - \{\xi_{c_0, R_0}\}$) and all $\delta \in \Delta(\xi)$ one has $\partial_-\mathcal{R}_\delta(\xi) \notin]-\nu_{\mathcal{R}}, \nu_{\mathcal{R}}[-\{0\}$ (resp. $\partial_+\mathcal{R}(\xi) \notin]-\nu_{\mathcal{R}}, \nu_{\mathcal{R}}[-\{0\}$).
- (C5) $\Gamma(\mathcal{R})$ is directionally finite at all its bifurcation points i.e. $\Delta(\xi, \Gamma(\mathcal{R}))$ is finite for all bifurcation point $\xi \in \Gamma(\mathcal{R})$. If (C5) holds we will say that \mathcal{R} is *directionally finite*.
- (C6) There exists a finite set $\mathcal{C}(\mathcal{R}) \subseteq X$ such that if $\xi \in \Gamma(\mathcal{R}) - \mathcal{C}(\mathcal{R})$ is a bifurcation point of $\Gamma(\mathcal{R})$ not in the Shilov boundary of X , then \mathcal{R} is super-harmonic at ξ (cf. Def. 2.10 below).

DEFINITION 2.10. *Let $\xi \in X_{\text{int}}$ not belonging to the Shilov boundary of X . Assume that $\partial_+\mathcal{R}(\xi)$ and $\partial_-\mathcal{R}_\delta(\xi)$ exists for all $\delta \in \Delta(\xi)$. We say that \mathcal{R} is super-harmonic (resp. sub-harmonic; harmonic) at ξ if the directions $\delta \in \Delta(\xi)$ such that $\partial_-\mathcal{R}_\delta(\xi) \neq 0$ are finite in number, and one has*

$$\partial_+\mathcal{R}(\xi) \leq \sum_{\delta \in \Delta(\xi)} \partial_-\mathcal{R}_\delta(\xi) \quad (2.8)$$

(resp. one has \geq ; equality holds). We say that \mathcal{R} is super-harmonic (resp. sub-harmonic; harmonic) if it is super-harmonic at all point $\xi \in X - \Gamma_X$. We call Laplacian of \mathcal{R} at ξ the negative number $\text{dd}^c(\mathcal{R}, \xi) := \partial_+\mathcal{R}(\xi) - \sum_{\delta \in \Delta(\xi, \Gamma(\mathcal{R}))} \partial_-\mathcal{R}_\delta(\xi)$.

REMARK 2.11. *In the definition one has to exclude the points of the Shilov boundary of X because polynomials in $K[T]$ are not sub-harmonic at such points, indeed some directions are removed.*

REMARK 2.12. *Below we prove that if \mathcal{R} verifies (C1),(C2),(C3),(C5), then the sum (2.8) is automatically finite. In fact if $\xi \notin \Gamma(\mathcal{R})$, then the sum is trivially verified with $0 \leq \sum_{\delta \in \Delta(\xi)} 0$. And if $\xi \in \Gamma(\mathcal{R})$, then in the sum one can replace $\Delta(\xi)$ by the finite set $\Delta(\xi, \Gamma(\mathcal{R}))$. Indeed by Prop. 2.17 the directions $\delta \in \Delta(\xi) - \Delta(\xi, \Gamma(\mathcal{R}))$ are all flat for \mathcal{R} i.e. $\partial_-\mathcal{R}_\delta(\xi) = 0$ (cf. Def. 2.9).*

REMARK 2.13. *Definition 2.10 is less general with respect to the usual definition of super-harmonicity, as for example those in [BR10], [Thu05], [FJ04]. The general definition allows an infinite number direction of non zero slope and the finite sum of (2.8) is replaced by an infinite one.*

THEOREM 2.14. *If $\mathcal{R} : X \rightarrow \mathbb{R}_{>0}$ satisfies the six conditions (C1)–(C6), then \mathcal{R} is finite.*

Proof. Since Γ is finite we are reduced to prove that $\Gamma' := \Gamma(\mathcal{R}) \cup \Gamma$ is finite. Since $\Gamma(\mathcal{R})$ is directionally finite at its bifurcation points, it is enough to prove that there are a finite number of bifurcation points of Γ' . The points in $\mathcal{C} := \mathcal{C}(\mathcal{R}) \cup \mathcal{C}(\Gamma)$ are finite in number and we can neglect them (cf. Def. 2.8). Moreover up to replace Γ by $\Gamma \cup \text{Sat}(\mathcal{C})$ we can assume $\mathcal{C}(\mathcal{R}) \subset \Gamma$. We distinguish the points of Γ from those in $\Gamma' - \Gamma$. Each point $\xi \in \Gamma - \mathcal{C}$ is a smooth point of Γ which is not punctured smooth. Hence $\xi \in \Gamma - \mathcal{C}$ is a bifurcation point of Γ' if and only if there exists a direction δ through ξ belonging to Γ' but not to Γ (i.e. $\delta \in \Delta(\xi, \Gamma') - \Delta(\xi, \Gamma)$). A representative branch $\Lambda(\xi_t)$ for δ defines a non generic open disk $D^-(t, \rho_\Gamma(t))$ which is tangent to Γ at $\xi = \xi_{t, \rho_\Gamma(t)}$ and which intersects Γ' . The proof is then divided in two parts. Firstly we prove that there are finitely many bifurcation points of Γ' belonging to $\Gamma - \mathcal{C}$ (cf. Proposition 2.18). This amounts to prove that there are finitely many disks of the above type. Secondly we prove that inside each disk tangent to Γ there are finitely many bifurcation points of Γ' (cf. Proposition 2.19).

LEMMA 2.15 (Flat directions do not belongs to the skeleton of \mathcal{R}). *Let $\xi \in X$, let t be a Dwork generic point for ξ and let $\rho \leq \rho_\Gamma(t)$. Assume that \mathcal{R} satisfies for all $\xi' \in D^-(t, \rho)$ the conditions (C1)': $\rho_{\mathcal{R}}(\xi') > 0$ and (C3)': ${}^L\mathcal{R}_{\xi'}$ is concave on $] - \infty, \ln(\rho)[$. Then ${}^L\mathcal{R}_\xi$ is non constant along the segment $] - \infty, \ln(\rho)[$ if and only if $\rho_{\mathcal{R}}(\xi) < \rho$, and in this case ${}^L\mathcal{R}_\xi$ has a break at $\tau := \ln(\rho_{\mathcal{R}}(\xi))$.*

Proof. If ${}^L\mathcal{R}_t$ is not constant on $] - \infty, \ln(\rho)[$, then \mathcal{R} is non constant in $D^-(t, \rho)$, so $\rho_{\mathcal{R}}(t) < \rho$. Conversely assume that $\rho_{\mathcal{R}}(t) < \rho$ and, by contrapositive, that ${}^L\mathcal{R}_t$ is constant on $] - \infty, \tau']$ for some $\tau < \tau' < \ln(\rho)$. Let $\rho' := \exp(\tau') > \rho_{\mathcal{R}}(t)$. Since \mathcal{R} is not constant on $D^-(t, \rho')$, there exists $\Omega' \in E(\Omega)$ and $t' \in D_{\Omega'}^-(t, \rho')$ such that $\mathcal{R}(\xi_{t'}) \neq \mathcal{R}(\xi_t)$. We now consider ${}^L\mathcal{R}_{t'}$. For all $\rho \geq |t - t'|$ one has $\mathcal{R}_t(\rho) = \mathcal{R}_{t'}(\rho)$ because $\lambda_{\xi_t}(\rho) = \lambda_{\xi_{t'}}(\rho)$. So $\mathcal{R}_{t'}$ is constant and equal to \mathcal{R}_t along $] |t - t'|, \rho']$ with value $\mathcal{R}(\xi_t)$. This contradicts the concavity of ${}^L\mathcal{R}_{t'}$ along $] - \infty, \ln(\rho)[$ because, by (C1)', ${}^L\mathcal{R}_{t'}$ is also constant in a neighborhood of $-\infty$ with value $\ln(\mathcal{R}(\xi_{t'})) \neq \ln(\mathcal{R}(\xi_t))$. \square

REMARK 2.16. *A open disk $D^-(t, \rho)$ is generic if and only if $\rho \leq r(\xi_t)$. The left slope of ${}^L\mathcal{R}_t$ along λ_t can be defined also at $\tau := \ln(\rho) \leq \ln(r(\xi_t))$, but it is always zero since \mathcal{R} is constant on each generic disk (cf. (2.1)). In other words generic disks define flat directions.*

PROPOSITION 2.17. *Assume that \mathcal{R} satisfies (C1), (C2), (C3). Let $D^-(t, \rho)$ be a non generic disk which is tangent to Γ at $\xi = \xi_{t, \rho}$. Let $\delta \in \Delta(\xi)$ be the direction defined by $D^-(t, \rho)$. Then $D^-(t, \rho)$ intersects Γ' (or equivalently $\Gamma(\mathcal{R})$) if and only if δ is not a flat direction i.e. if $\partial_- \mathcal{R}_\delta(\xi) < 0$ (cf. Def. 2.9).*

Proof. Conditions (C1), (C2) and (C3) imply that the slopes of ${}^L\mathcal{R}_t$ along $]0, \rho[$ are all negatives or equal to zero. Since $\rho_{\mathcal{R}}(t) = \rho_{\Gamma(\mathcal{R})}(t)$ (cf. section 2.0.2), then $D^-(t, \rho)$ intersects $\Gamma(\mathcal{R})$ if and only if $\rho_{\mathcal{R}}(t) < \rho$. Lemma 2.15 implies $\partial_- \mathcal{R}_\delta(\xi) < 0$ because ${}^L\mathcal{R}_t$ has at least a break at $\rho_{\mathcal{R}}(\xi) < \rho$. \square

PROPOSITION 2.18. *There are a finite number of bifurcation points of Γ' belonging to Γ .*

Proof. It is enough to prove that along each individual maximal branch $\Lambda(\xi)$ of Γ there is a finite number of bifurcation points ξ of $\Gamma(\mathcal{R})$. As observed the points of \mathcal{C} are finite in number and we can neglect them. By Proposition 2.17 the super-harmonicity (C6) implies that ${}^L\mathcal{R}_\xi : [r(\xi), R_0] \rightarrow \mathbb{R}$ has a break at each bifurcation point of $\Gamma(\mathcal{R})$ belonging to $\Lambda(\xi) - \mathcal{C}$. By (C3) there are a finite number of breaks along $\Lambda(\xi)$, and hence a finite number of bifurcation points of $\Gamma(\mathcal{R})$. \square

PROPOSITION 2.19. *There is a finite number N of bifurcation points of $\Gamma(\mathcal{R})$ inside a given (non generic) disk $D^-(t, \rho_\Gamma(t))$ which is tangent to Γ . Moreover, since $\mathcal{C}(\mathcal{R}) \subset \Gamma$, then $0 \leq N \leq \frac{-\partial_- \mathcal{R}_\delta(\xi)}{2\nu_{\mathcal{R}}}$, where $\xi = \xi_{t, \rho_\Gamma(t)}$ and $\delta \in \Delta(\xi)$ is the direction defined by the disk $D^-(t, \rho_\Gamma(t))$.*

Proof. Let $\text{Bif}(\Gamma(\mathcal{R})) \subseteq \Gamma(\mathcal{R})$ be the set of bifurcation points of $\Gamma(\mathcal{R})$. For all $\xi' \in \text{Bif}(\Gamma(\mathcal{R}))$ let $N(\xi')$ denote the number of directions through ξ' belonging to $\Gamma(\mathcal{R})$ i.e. $N(\xi')$ equals the cardinality of $\Delta(\xi', \Gamma(\mathcal{R}))$ which is finite by (C5). By (C6) and (C4), if $\xi' \in \text{Bif}(\Gamma(\mathcal{R})) \cap D^-(t, \rho_\Gamma(t))$ one has

$$\partial_+ \mathcal{R}(\xi') \leq \sum_{\delta' \in \Delta(\xi', \Gamma(\mathcal{R}))} \partial_- \mathcal{R}_{\delta'}(\xi') \leq -\nu_{\mathcal{R}} \cdot N(\xi') \leq -2\nu_{\mathcal{R}} < 0. \quad (2.9)$$

In particular the function ${}^L\mathcal{R}_t$ has a break in correspondence to each bifurcation point of $\Gamma(\mathcal{R})$ in $\Lambda(\xi_t)$. At each break point along $\Lambda(\xi_t)$ one has (2.9) hence $\partial_- \mathcal{R}_\delta(|\cdot|) \leq -2\nu_{\mathcal{R}} \cdot N_t$, where N_t is the number of bifurcation points along $\Lambda(\xi_t)$. If by contrapositive one has an infinite number of bifurcation points in $D^-(t, \rho_\Gamma(t))$, then for all integer n there exists a branch $\Lambda(\xi_{t_n})$ with t_n in $D^-(t, \rho_\Gamma(t))$ having at least n bifurcation points of $\Gamma(\mathcal{R})$. This forces $\partial_- \mathcal{R}_\delta(\xi)$ to be less than or equal to $-2\nu_{\mathcal{R}} \cdot n$ for all n , which is absurd because $\partial_- \mathcal{R}_\delta(\xi) > -\infty$. So the number of bifurcation points of $\Gamma(\mathcal{R})$ inside $D^-(t, \rho_\Gamma(t))$ is finite. \square

This completes the proof of theorem 2.14. \square

DEFINITION 2.20. Let $A(t, I) := \{|T - t| \in I\}$ be a possibly not closed annulus or disk (if $0 \in I$ one has a disk), and let $\mathcal{R} : A(t, I) \rightarrow \mathbb{R}$ be a function. We say that \mathcal{R} is finite over $A(t, I)$ if there exists a compact sub-interval $J \subset I$ such that $\Gamma(\mathcal{R}|_J)$ is finite over $A(t, J)$, and for all compact $J \subseteq J' \subseteq I$ one has $\Gamma(\mathcal{R}|_{J'}) = \Gamma(\mathcal{R}|_J) \cup \Gamma_{A(t, J')}$ over $A(t, J')$. In this case we define the skeleton $\Gamma(\mathcal{R})$ over $A(t, I)$ as $\Gamma(\mathcal{R}) := \Gamma(\mathcal{R}|_J) \cup \{\xi_{0, \rho}\}_{\rho \in I}$.

COROLLARY 2.21. Let \mathcal{R} be a function defined on a possibly not closed annulus or disk $A(t, I) := \{|T - t| \in I\}$ (if $0 \in I$ one has a disk). Fix once for all two finite sets $S, \mathcal{C}(\mathcal{R}) \subset X$, and set $\Gamma = \begin{cases} \text{Sat}(S) \cup \{\xi_{t, \rho}\}_{\rho \in I} & \text{if } 0 \notin I \\ \text{Sat}(S) & \text{if } 0 \in I \end{cases}$. Impose that $\Gamma_X \subseteq \Gamma$ (i.e. S is not empty) if $I = [0, R]$. Assume that \mathcal{R} verifies the six properties (C1)–(C6) with respect to Γ and $\mathcal{C}(\mathcal{R})$ on each sub-annulus (resp. sub-disk) $A(t, J)$, with J compact, $J \subseteq I$ (resp. $0 \in J$ if $0 \in I$). Assume moreover that \mathcal{R} has a finite number of breaks along $\{\xi_{t, \rho}\}_{\rho \in I}$. Then $\Gamma(\mathcal{R})$ is continuous, finite and it factorizes through $\Gamma(\mathcal{R})$. Moreover if $A(t, I)$ is an open disk, if Γ and $\mathcal{C}(\mathcal{R})$ are both empty, and if s is the last slope of ${}^L\mathcal{R}_t$ as in Prop. 2.19, then the number N of bifurcation points of $\Gamma(\mathcal{R})$ verifies $0 \leq N \leq \frac{-s}{2\nu_{\mathcal{R}}}$.

Proof. By Thm. 2.14 the restriction of \mathcal{R} is finite on each closed sub-annulus $\{|T - t| \in J\} \subseteq \{|T - t| \in I\}$. By super-harmonicity, each new branch of $\Gamma(\mathcal{R})$ generates a break along $\{\xi_{t, \rho}\}_{\rho \in I}$. \square

EXAMPLE 2.22. 1. The function $\xi \mapsto \rho_{\xi, X}$ verifies the six properties (C1)–(C6) with $\Gamma = \Gamma_X$, and $\mathcal{C}(\rho_{-, X}) = \emptyset$. It is moreover super-harmonic in the sense of definition 2.10, and $\Gamma(\rho_{-, X}) = \Gamma_X$.

2. If $\mathcal{R}_1, \dots, \mathcal{R}_n$ are functions satisfying the six properties (C1)–(C6), then so does $\min(\mathcal{R}_1, \dots, \mathcal{R}_n)$.

3. Let $f_1, \dots, f_n \in \mathcal{O}(X)$ and $\alpha_1, \dots, \alpha_n > 0$. Assume that each f_i has no zeros on $\mathcal{O}(X)$. Then the function $\mathcal{R}(\xi) := \min_i |f_i(\xi)|^{-\alpha_i}$ verifies (C1)–(C6), with $\Gamma = \Gamma_X$, and $\Gamma(\mathcal{R}) = \Gamma_X$. If $K = \widehat{K}^{\text{alg}}$, then \mathcal{R} is also super-harmonic (cf. Def. 2.10) because so does each function $\xi \mapsto |f_i(\xi)|^{-\alpha_i}$.

2.4 Permanence of (C1)–(C6) by descent of the ground field.

For $\Omega \in E(K)$ let $\text{Pr}_K^\Omega : X \widehat{\otimes} \Omega \rightarrow X$ be the canonical projection, and let $\mathcal{R}' = \mathcal{R} \circ \text{Pr}_K^\Omega$. For all $\Omega' \in E(\Omega)$ and all $t \in X(\Omega')$ one obviously have $\rho_{\mathcal{R}}(t) = \rho_{\mathcal{R}'}(t)$, because $\rho_{\mathcal{R}}(t)$ only depend on $X(\Omega')$ for an unspecified $\Omega' \in E(K)$ (cf. (2.2)). This immediately gives $\Gamma(\mathcal{R}) = \text{Pr}_K^\Omega(\Gamma(\mathcal{R}'))$. The finiteness and the directional finiteness (C5) of $\Gamma(\mathcal{R}')$ implies then that of $\Gamma(\mathcal{R})$. Moreover the restrictions \mathcal{R}_t and \mathcal{R}'_t coincide as functions on $[0, R_0]$ (cf. (2.5)). So the first four properties (C1)–(C4) hold for \mathcal{R}' if and only if they hold for \mathcal{R} . Conversely the super-harmonicity of \mathcal{R}' over X' does not imply that of \mathcal{R} . The problem arises at the bifurcation points of $\Gamma(\mathcal{R}) \cap \Gamma$, in particular at those of Γ_X . This is the reason for which one introduces property (C6) instead of the full super-harmonicity. In

sections 2.4.1 and 2.4.2 below we prove that if \mathcal{R}' satisfies (C6) then so does \mathcal{R} . We firstly analyze the case $\Omega = \widehat{K^{\text{alg}}}$ (cf. section 2.4.1) and then the general case (cf. section 2.4.2).

REMARK 2.23. If $K = \widehat{K^{\text{alg}}}$ the functions in $\mathcal{O}(X)$ are harmonic at each point $\xi \in X_{\text{int}} - \Gamma_X$. If $K \neq \widehat{K^{\text{alg}}}$, then the functions in $\mathcal{O}(X)$ are not all super-harmonic nor all sub-harmonic. As a counterexample consider $p = 3$, $K = \mathbb{Q}_p$, $f(T) := (T - 1)^p/(T - 1)$, with roots α_1, α_2 , and $X \widehat{\otimes} \widehat{K^{\text{alg}}} = D^+(0, 1)$ with holes $D^-(\alpha_i, \varepsilon)$, $0 < \varepsilon < |\alpha_1 - \alpha_2|$. Then $f, f^{-1} \in \mathcal{O}(X)$, and the two paths λ_{α_1} and λ_{α_2} are identified in X by $\text{Gal}(K^{\text{alg}}/K)$. So $\Gamma(f) = \Gamma(f^{-1}) = \Lambda(\xi_{\alpha_i, \varepsilon})$ in X . f and f^{-1} are both harmonic as functions on $X \widehat{\otimes} \widehat{K^{\text{alg}}}$. While on X one sees that f (resp. f^{-1}) is log-convex (resp. log-concave) along λ_{α_i} , hence sub-harmonic (resp. super-harmonic) at $\xi_{\alpha_i, |\alpha_1 - \alpha_2|}$.

2.4.1 Algebraic extensions. Let $X' := X \widehat{\otimes} \widehat{K^{\text{alg}}}$. By [Ber90, 1.3.6] the canonical map $\text{Pr}_K^{\widehat{K^{\text{alg}}}}$ identifies X with $X'/\text{Gal}(K^{\text{alg}}/K)$. Let $\Gamma' := (\text{Pr}_K^{\widehat{K^{\text{alg}}}})^{-1}(\Gamma)$ and $\mathcal{C}(\mathcal{R}') := (\text{Pr}_K^{\widehat{K^{\text{alg}}}})^{-1}(\mathcal{C}(\mathcal{R}))$. Assume that \mathcal{R}' verifies the six properties (C1)–(C6) with respect to Γ' and $\mathcal{C}(\mathcal{R}')$. Let $\xi' \in X'$, and let $\xi := \text{Pr}_K^{\widehat{K^{\text{alg}}}}(\xi')$. If a set of directions $\delta_1, \dots, \delta_{n_\delta} \in \Delta(\xi')$ form an orbit under $\text{Gal}(K^{\text{alg}}/K)$ and if δ is the corresponding direction of ξ , then as observed one has $\partial_+ \mathcal{R}'(\xi') = \partial_+ \mathcal{R}(\xi)$ and $\partial_- \mathcal{R}'_{\delta_i}(\xi') = \partial_- \mathcal{R}'_{\delta_i}(\xi')$ for all $i = 1, \dots, n_\delta$. We call n_δ the multiplicity of δ . Replacing each $\partial_- \mathcal{R}'_{\delta_i}(\xi')$ by $\partial_- \mathcal{R}_\delta(\xi)$ in the formula (2.8) one sees that the contribution $\partial_- \mathcal{R}_\delta(\xi)$ of δ to the super-harmonicity of \mathcal{R} at ξ equals $\frac{1}{n_\delta} \cdot S'$, where $S' := \sum_{i=1}^{n_\delta} \partial_- \mathcal{R}'_{\delta_i}(\xi')$ is the contribution of the orbit $\{\delta_i\}_i$ to the super-harmonicity of \mathcal{R}' at ξ' . If $\delta_i \notin \Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}'))$, then $\partial_- \mathcal{R}'_{\delta_i}(\xi') \leq 0$ for all i (cf. Proposition 2.17) so that $S' \leq \partial_- \mathcal{R}_\delta(\xi)$. Hence if $\xi' \notin \Gamma(\mathcal{R}') \cap \Gamma'$ the super-harmonicity of \mathcal{R}' at ξ' implies that of \mathcal{R} at ξ . The same holds if $\xi' \in \Gamma(\mathcal{R}') \cap \Gamma'$ is not a bifurcation point of $\Gamma(\mathcal{R}') \cap \Gamma'$ because in this case there are three kind of directions: $\Delta(\xi', \Gamma(\mathcal{R}') \cap \Gamma') = \{\delta\}$ is reduced to a single element, so $n_\delta = 1$ and its contribution to the super-harmonicity is the same over K or $\widehat{K^{\text{alg}}}$, the directions $\delta \in \Delta(\xi', \Gamma(\mathcal{R}'))$ satisfy $S' \leq \partial_- \mathcal{R}_\delta(\xi)$ as above, and finally the other directions does not contribute to the super-harmonicity. The problem arises if $\xi' \in \Gamma(\mathcal{R}') \cap \Gamma'$ is a bifurcation point of $\Gamma(\mathcal{R}') \cap \Gamma'$. In this case $\partial_- \mathcal{R}'_{\delta}(\xi')$ can be positive for $\delta \in \Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}'))$, and the super-harmonicity of \mathcal{R}' at ξ' does not implies the super-harmonicity of \mathcal{R} at ξ unless the multiplicity of each direction in $\Delta(\xi', \Gamma' \cap \Gamma(\mathcal{R}'))$ is equal to one i.e. if $\Gamma(\mathcal{R}') \cap \Gamma' = \text{Sat}(\{\xi_{t_1, \rho_1}, \dots, \xi_{t_n, \rho_n}\})$ with $t_1, \dots, t_n \in X(K)$, in other words if Γ' is K -rational (cf. section 1.4). This proves the following

PROPOSITION 2.24. If \mathcal{R}' verifies the five properties (C1)–(C5) then so does \mathcal{R} . If one choses $\mathcal{C}(\mathcal{R})$ in order that $\mathcal{C}(\mathcal{R}')$ contains the bifurcation points of $\Gamma(\mathcal{R}') \cap \Gamma'$, then property (C6) descends from \mathcal{R}' to \mathcal{R} . If moreover $\Gamma \cap \Gamma(\mathcal{R})$ is K -rational and if \mathcal{R}' is super-harmonic, then so does \mathcal{R} (Note that if Γ is K -rational then so does $\Gamma \cap \Gamma(\mathcal{R})$). \square

2.4.2 Transcendental extensions. Let $K = \widehat{K^{\text{alg}}}$, let $\Omega \in E(K)$, and $X' := X \widehat{\otimes}_K \Omega$. Let $\xi' \in \Gamma(\mathcal{R}')$ and let $\xi \in \Gamma(\mathcal{R})$ be its image. Super-harmonicity concerns bifurcation points. They are all of type (2) in the sense of [Ber90, 1.4.4]. By Remark 1.10 the directions in $\Delta(\xi)$ and in $\Delta(\xi')$ all admit a representative branch of the type $\Lambda(\xi_t)$, with $t \in X(K) \subset X(\Omega)$. One has the following commutative diagram (cf. (1.6)) where both i_K and i_Ω are injective maps:

$$\begin{array}{ccc} X(\Omega) & \xrightarrow{i_\Omega} & X' \\ \cup & & \downarrow \text{Pr}_K^\Omega \\ X(K) & \xrightarrow{i_K} & X. \end{array} \quad (2.10)$$

Each path $\lambda_{\xi_t} : [0, R_0] \rightarrow X$, with $t \in X(K) \subset X(\Omega)$, admits then a canonical lifting $\lambda_{\tilde{\xi}_t} : [0, R_0] \rightarrow X'$, where $\tilde{\xi}_t = i_\Omega(t) \in X'$. One hence has a canonical injective map $\Delta(\xi) \subseteq \Delta(\xi')$. Note that there

is no map $\Delta(\xi') \rightarrow \Delta(\xi)$ corresponding to Pr_K^Ω . In fact if a direction $\delta' \in \Delta(\xi')$ is defined by a disk $D^-(t', \rho)$ which is generic over K , but not over Ω , then the image in X of $D^-(t', \rho)$ is reduced to the point $\{\xi = \xi_{t', \rho}\}$, and there is no directions in $\Delta(\xi)$ corresponding to δ' . In this case we say that δ' is *contracted* to ξ by Pr_K^Ω . This is in fact the case for all directions $\delta' \in \Delta(\xi') - \Delta(\xi)$. Indeed Pr_K^Ω identifies $\Gamma_{X'}$ with Γ_X (which is explicitly given in both cases by $\text{Sat}(\{\xi_{c_i, R_i}\}_{i=0, \dots, \mu})$ cf. section 1.5) so one has the equality $\Delta(\xi', \Gamma_{X'}) = \Delta(\xi, \Gamma_X)$. On the other hand if δ' does not belong to $\Gamma_{X'}$ then it is defined by a disk $D^-(t', \rho)$ which is non generic over Ω . Clearly $D^-(t', \rho)$ is non generic over K if and only if $\delta' \in \Delta(\xi)$. So each direction in $\Delta(\xi') - \Delta(\xi)$ is contracted to ξ . These directions are not in $\Gamma(\mathcal{R}')$ since defined by generic disks on which \mathcal{R}' is constant.

PROPOSITION 2.25. *Let $\xi' \in \Gamma(\mathcal{R}')$, and let $\xi \in \Gamma(\mathcal{R})$ be its image in X . Then*

$$\Delta(\xi, \Gamma(\mathcal{R})) = \Delta(\xi', \Gamma(\mathcal{R}')) . \quad (2.11)$$

Hence the super-harmonicity is preserved by extending or descending the scalars from K to Ω . \square

REMARK 2.26. *The radius of convergence function \mathcal{R}^M will satisfy the six conditions (C1)–(C6) with respect to $\Gamma := \Gamma_X$ and $\mathcal{C}(\mathcal{R}) := \mathcal{C}_X$, and it will be also super-harmonic over any $\Omega \in E(\widehat{K^{\text{alg}}})$. If X is K -rational, then \mathcal{R}^M will be super-harmonic on X (cf. Prop. 2.24)*

3. Radius of convergence function of an ultrametric differential module

Thank to 2.4 one can assume $K = \widehat{K^{\text{alg}}}$. Nevertheless we do not apply systematically this assumption in order to point out the problems over K (e.g. Lemma 3.4 and super-harmonicity in general).

3.1 Preliminaries

A *differential ring* is a ring A together with a derivation $d : A \rightarrow A$. A *differential module* over (A, d) is a finite free A -module together with a linear map $\nabla : M \rightarrow M$, called the *connection* of M , satisfying $\nabla(am) = d(a)m + a\nabla(m)$ for all $a \in A$ and $m \in M$. The choice of a basis of M provides an isomorphism of A modules $M \xrightarrow{\sim} A^r$, and the operator ∇ is given in this basis by the rule

$$\nabla(a_1, \dots, a_r)^t = (d(a_1), \dots, d(a_r))^t - G \cdot (a_1, \dots, a_r)^t \quad (3.1)$$

where $G \in M_r(A)$ is a matrix. Reciprocally the data of such a matrix defines a differential module structure on A^r by the rule (3.1). A morphism between differential modules is an A -linear map $M \rightarrow N$ commuting with the connections. We denote by $d\text{-Mod}(A)$ the category of differential modules over A . Because we use Taylor solutions in the sequel the derivation will always be d/dT , and $A = \mathcal{O}(X)$, $\mathcal{H}(\xi)$, $\mathcal{A}_\Omega(t, \rho)$, ... If $A = \mathcal{O}(X)$ we will often restrict M to a sub-affinoid $X' \subseteq X$ or to a sub-disk $D^-(t, \rho) \subseteq X$. This means that we consider the scalar extension module $M \otimes_{\mathcal{O}(X)} \mathcal{O}(X')$ (resp. $M \otimes_{\mathcal{O}(X)} \mathcal{A}_\Omega(t, \rho)$) together with the connection $\nabla' := \nabla \otimes \text{Id} + \text{Id} \otimes d/dT$. A solution of M with values in $\mathcal{A}_\Omega(t, \rho)$ is an element of the kernel of ∇' acting on $M \otimes_{\mathcal{O}(X)} \mathcal{A}_\Omega(t, \rho)$. If we chose a basis as above (cf. (3.1)), such a solution $\vec{y} \in \mathcal{A}_\Omega(t, \rho)^r$ verifies $\vec{y}' = G(T) \cdot \vec{y}$. The Taylor solution of M at t is given by $Y(T, t) = \sum_{s \geq 0} G_s(t) \frac{(T-t)^s}{s!}$, where G_s is the matrix defined by $d^s(Y) = G_s(T) \cdot Y$. Namely $G_0 := \text{Id}$, $G_1 := G$, and recursively one has $G_{s+1} := d(G_s) + G_s \cdot G$ for all $s \geq 0$. The radius of convergence of this power series is given by

$$\mathcal{R}_\Omega^Y(t) := \liminf_s |G_s(t)/s!|_\Omega^{-1/s} \quad (3.2)$$

where if $G_s = (g_{s,i,j})$, then $|G_s(t)| = \max_{i,j} |g_{s,i,j}(t)|$. It is well known that $\mathcal{R}_\Omega^Y(t) > 0$ for all $t \in X(\Omega)$ (cf. [DGS94, Appendix III]). $\mathcal{R}_\Omega^Y(t)$ only depend on the Berkovich point $\xi_t \in X$ defined by t , and so one has a well defined function $\mathcal{R}^Y : X \rightarrow \mathbb{R}_{>0} \cup \{+\infty\}$.

DEFINITION 3.1. The radius of convergence function of M is the function $\mathcal{R}^M : X \rightarrow]0, R_0]$ associating to each $\xi \in X$ the positive real number

$$\mathcal{R}^M(\xi) := \min(\mathcal{R}^Y(\xi), \rho_{\xi, X}). \quad (3.3)$$

The spectral radius of convergence function $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}} : X \rightarrow [0, R_0]$ of M is defined by

$$\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}(\xi) := \min(\mathcal{R}^Y(\xi), r(\xi)).^9 \quad (3.4)$$

The notation $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}$ will be justified in Remark 3.9. As for \mathcal{R}^Y one always has $\mathcal{R}^M(\xi) > 0$ (and also $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}(\xi) > 0$ if $r(\xi) > 0$). The matrices $Y(T, t)$ and G_s depend on a basis of M , as well the function \mathcal{R}^Y . On the other hand the presence of $\rho_{\xi, X}$ and $r(\xi)$ in the definitions of \mathcal{R}^M and $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}$ respectively makes them invariant under base changes of M . The Taylor solution matrix $Y(T, t)$ lies then in $GL_r(\mathcal{A}_\Omega(t, \mathcal{R}^M(\xi_t)))$, and it verifies $Y(T, t) = Y(T, t') \cdot Y(t', t)$ for all t' satisfying $|t - t'| < \mathcal{R}^M(\xi_t)$ (cf. [Chr12]). In particular the radius of convergence $\mathcal{R}^Y(\xi_t)$ of $Y(T, t)$ is larger than or equal to the radius of convergence $\mathcal{R}^Y(\xi_{t'})$ of $Y(T, t')$, by symmetry¹⁰ the two radii are actually equal $\mathcal{R}^Y(\xi_t) = \mathcal{R}^Y(\xi_{t'})$. This fact together with (2.1) imply

$$\max(\mathcal{R}^M(\xi), r(\xi)) \leq \rho_{\mathcal{R}^M}(\xi), \quad (3.5)$$

$$\rho_{\mathcal{R}^Y}(\xi) = \rho_{\mathcal{R}^M}(\xi), \quad (3.6)$$

$$\rho_{\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}}(\xi) = r(\xi). \quad (3.7)$$

From (3.6) one immediately has

$$\Gamma(\mathcal{R}^M) = \Gamma(\mathcal{R}^Y) \quad (3.8)$$

and from (3.7) one has $\Gamma(\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}) = \Gamma(r) = X$ (here $r : \xi \mapsto r(\xi)$ cf. Def. 1.10). So $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}$ is not finite. From (3.5) and (3.6) we deduce that if $\rho_{\mathcal{R}^M}(\xi) = r(\xi)$ (i.e. if $\xi \in \Gamma(\mathcal{R}^M)$), then

$$\mathcal{R}^M(\xi) = \mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}(\xi) \leq r(\xi), \quad (3.9)$$

and if the inequality is strict, then $\mathcal{R}^M(\xi) = \mathcal{R}^{M \widehat{\otimes} K^{\text{alg}, \text{sp}}}(\xi) = \mathcal{R}^Y(\xi)$. From (3.5) and (2.1) one also obtains the following often useful expression of \mathcal{R}^M :

$$\mathcal{R}^M(\xi) = \min(\mathcal{R}^Y(\xi), \rho_{\mathcal{R}^Y}(\xi)). \quad (3.10)$$

Another important property of \mathcal{R}^M and \mathcal{R}^Y is the so called *transfer theorem* which affirms that they inverse the natural partial order of X , that is : if $\xi_1(f) \leq \xi_2(f)$ for all $f \in \mathcal{O}(X)$, then

$$\mathcal{R}^Y(\xi_1) \geq \mathcal{R}^Y(\xi_2) \quad \text{and} \quad \mathcal{R}^M(\xi_1) \geq \mathcal{R}^M(\xi_2). \quad (3.11)$$

For \mathcal{R}^Y this immediately follows from (3.2). For \mathcal{R}^M this follows from (3.3) since one has $\rho_{\xi_1, X} = \rho_{\xi_2, X}$ (cf. section 1.5). The transfer theorem can be rephrased as follows. If $0 \leq \rho \leq \rho_{\xi, X}$, then $\mathcal{R}^M(\xi) \geq \mathcal{R}^M(\lambda_\xi(\rho))$ i.e. \mathcal{R}_ξ^M is decreasing on $[0, \rho_{\xi, X}]$. Note that ${}^L\mathcal{R}_\xi^M$ is moreover concave on $] - \infty, \ln(\rho_{\xi, X})]$, because by (3.2) it is \liminf of concave functions. More generally if I is an interval with interior $\overset{\circ}{I}$ and if the annulus $\{|T - t_\xi| \in \overset{\circ}{I}\}$ is contained in X , then \mathcal{R}_ξ^M is log-concave on I by the same reason (cf. property (LC) of section 1.3).

REMARK 3.2. The function \mathcal{R}^Y and \mathcal{R}^M are invariant by scalar extension of the ground field K , hence all considerations of section 2.4 hold. More precisely let $\Omega \in E(K)$, $X' = X \widehat{\otimes} \Omega$, $M' := M \widehat{\otimes}_K \Omega$. Let \mathcal{R}'^Y be the function (3.2) on X' defined by M' in the basis $e \otimes 1$, where $e \subset M$ is the basis that serves to define \mathcal{R}^Y . Then $\mathcal{R}'^Y = \text{Pr}_K^\Omega \circ \mathcal{R}^Y$ since the matrices $\{G_s\}_s$ of (3.2) are the same in

⁹The spectral radius is often called *generic radius* by the authors, e.g. [Ked10b].

¹⁰ $Y(t, t') \in GL_n(\Omega)$ is invertible with inverse $Y(t', t)$.

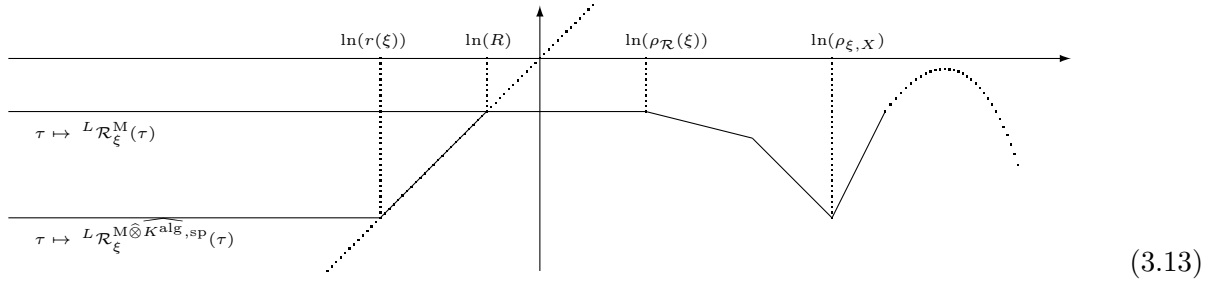
both cases. This immediately gives $\mathcal{R}^{M'} = \text{Pr}_K^\Omega \circ \mathcal{R}^M$ since $\rho_{-,X}$ is independent on Ω . Conversely $\xi \mapsto r(\xi)$ depends on K and so $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}$ is invariant if and only if Ω/K is algebraic.

3.2 Comparison between $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}$ and \mathcal{R}^M .

We now compare $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}$ and \mathcal{R}^M along a branch. Definition 3.1 immediately gives $\mathcal{R}^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\xi) = \min(r(\xi), \mathcal{R}^M(\xi))$, for all $\xi \in X$. Since $r(\lambda_\xi(\rho)) = \max(\rho, r(\xi))$ one finds

$$\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \min(\max(r(\xi), \rho), \mathcal{R}_\xi^M(\rho)) \quad (3.12)$$

for all $\rho \in [0, R_0]$, as in the following picture where $R := \mathcal{R}^M(\xi) > 0$:



In particular since $\rho \mapsto \mathcal{R}_\xi^M(\rho)$ is log-concave for $\rho \in [0, \rho_{\xi,X}]$ and $\rho \mapsto \max(r(\xi), \rho)$ is log-convex for all $\rho \in [0, R_0]$, then

- i) For all $\rho \in [R, R_0]$ one has $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \mathcal{R}_\xi^M(\rho)$.
- ii) If $R \leq r(\xi)$, then $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \mathcal{R}_\xi^M(\rho)$ for all $\rho \in [0, R_0]$.
- iii) If $R > r(\xi)$, then $\forall \rho \in [0, R]$ one has $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \max(\rho, r(\xi))$ and $\mathcal{R}_\xi^M(\rho) = \mathcal{R}^M(\xi) = R$.

Indeed for all $\rho \in [\rho_{\xi,X}, R_0]$ one has $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \mathcal{R}_\xi^M(\rho)$ because $\lambda_\xi(\rho) \in \Gamma_X$ and hence $r(\lambda_\xi(\rho)) = \rho_{\lambda_\xi(\rho), X}$. So the above equalities have to be proved for $\rho < \rho_{\xi,X}$, and in this case they follow by the above convexity/concavity argument.

REMARK 3.3. If $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho) = \mathcal{R}_\xi^M(\rho)$ then for all $\rho' \in [\rho, R_0]$ one has $\mathcal{R}_\xi^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}}(\rho') = \mathcal{R}_\xi^M(\rho')$.

3.3 Spectral radius and spectral norm of the connection.

We quickly recall some facts that are necessary for the correct understanding of this paper. Let $(F, |\cdot|_F) \in E(K)$ and let V be a finite dimensional vector space. A norm $|\cdot|_V$ on V compatible with $|\cdot|_F$ is a map $|\cdot|_V : V \rightarrow \mathbb{R}_{\geq 0}$ such that (i) $|v|_V = 0$ if and only if $v = 0$; (ii) $|v - v'|_V \leq \max(|v|_V, |v'|_V)$ for all $v, v' \in V$; (iii) $|fv|_V = |f|_F \cdot |m|_V$ for all $f \in F, v \in V$. If $T : V \rightarrow V$ is a bounded \mathbb{Z} -linear operator, define $|T|_V := \sup_{v \neq 0} |T(v)|_V / |v|_V$, and $|T|_{S_p, |\cdot|_F} := \lim_s |T^s|_V^{1/s}$. One proves that the limit exists, and that $|T|_{S_p, |\cdot|_F}$ only depends on $|\cdot|_F$ and not on the choice of $|\cdot|_V$ compatible with $|\cdot|_F$ (cf. [Ked10b, Def. 6.1.3]). Let $\omega := \lim_n |n!|^{1/n}$. If the restriction of $|\cdot|$ to \mathbb{Q} is p -adic (resp. trivial), then $\omega = |p|^{\frac{1}{p-1}}$ (resp. $\omega = 1$).

If $\xi \in X^{\text{gen}}$ the kernel of ξ is zero, and ξ is a norm on $\mathcal{O}(X)$. In this case $(\mathcal{H}(\xi), \xi) = (\mathcal{F}(X), \xi)^\wedge$ is the completion of the fraction field $\mathcal{F}(X)$ of $\mathcal{O}(X)$ with respect to the norm ξ . The following lemma proves that the derivation d/dT is continuous, and hence it extends by continuity to $\mathcal{H}(\xi)$.

LEMMA 3.4. Let $\xi \in X^{\text{gen}}$ and let $V := F := \mathcal{H}(\xi)$. Then $|(d/dT)^n|_{\mathcal{H}(\xi)} \leq \frac{|n!|}{r(\xi)^n}$. Assume that

$K = \widehat{K^{\text{alg}}}$ or, if $K \neq K^{\text{alg}}$, that $\xi = \lambda_c(\rho)$ with $c \in K$ and $\rho > 0$. Then

$$|(d/dT)^n|_{\mathcal{H}(\xi)} = \frac{|n!|}{r(\xi)^n} \quad \text{and} \quad \|d/dT\|_{S_p, \xi} = \frac{\omega}{r(\xi)}. \quad (3.14)$$

Proof. If $t \in X(\Omega)$ is a Dwork generic point for ξ , then the Taylor expansion at t gives an injective map $\mathcal{H}(\xi) \rightarrow \mathcal{A}_\Omega(t, r(\xi))$. The image of $f \in \mathcal{H}(\xi)$ is $\sum_{i \geq 0} f^{(i)}(t)(T-t)^i/i!$ and $\xi(f) = \xi_{t,0}(f) = \xi_{t,r(\xi)}(f) = \sup_{i \geq 0} |f^{(i)}(t)/i!| \cdot r(\xi)^i$. From this one easily has $|(d/dT)^n|_{\mathcal{H}(\xi)} \leq |n!|/r(\xi)^n$. Now we prove the converse inequality. Assume first that $K = \widehat{K^{\text{alg}}}$. For all $c \in K$ one has $|n!| = |(d/dT)^n(T-c)^n|_t \leq |(d/dT)^n|_{\mathcal{H}(\xi)}|T-c|_t^n$. Hence $|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/|t-c|_\Omega^n$. Since this holds for all $c \in K = \widehat{K^{\text{alg}}}$ one finds $|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/r(\xi)^n$, because $r(\xi) = \inf_{c \in \widehat{K^{\text{alg}}}} |t-c|_\Omega$. If $K \neq K^{\text{alg}}$, but $\xi = \xi_{c,\rho}$, with $c \in K$, then $r(\xi_{c,\rho}) = \rho$. The above computation for the individual polynomial $(T-c)^n$ gives $|(d/dT)^n|_{\mathcal{H}(\xi)} \geq |n!|/|t-c|_\Omega^n = |n!|/\rho^n = |n!|/r(\xi)^n$ (cf. Lemma 1.3). \square

REMARK 3.5. The assumptions of Lemma 3.4 are possibly superfluous. The assumption $K = \widehat{K^{\text{alg}}}$ will not be relevant since \mathcal{R}_i^M are insensitive to base change of the ground field K (cf. Remark 4.6).

DEFINITION 3.6. Let $\xi \in X^{\text{gen}}$, and let $M = M_\xi$ be a differential module over $(\mathcal{H}(\xi), d/dT)$. We set

$$\mathcal{R}^{M, \text{sp}}(\xi) := \omega \cdot \|\nabla\|_{S_p, \xi}^{-1}, \quad (3.15)$$

where $\|\nabla\|_{S_p, \xi}$ is the spectral norm of $\nabla : M_\xi \rightarrow M_\xi$ with respect to the norm $|\cdot|_F = \xi$ on $F := \mathcal{H}(\xi)$. One extends this definition to the whole X by setting $\mathcal{R}^{M, \text{sp}}(\xi) = 0$ for all $\xi \in X - X^{\text{gen}}$.

REMARK 3.7. $\mathcal{R}^{M, \text{sp}}(\xi)$ coincides with the spectral radius $R(V)$ studied in [Ked10b, Def.9.4.4] and [CD94, Section 2.3]. It is the radius studied in the whole literature (up to [BV07] and [Bal10]).

A direct computation gives (cf. [Ked10b, Lemma 6.2.5], [CD94, Prop.1.3])

$$\|\nabla\|_{S_p, \xi} = \max(\limsup_s |G_s(\xi)|^{1/s}, \|d/dT\|_{S_p, \xi}), \quad (3.16)$$

where G_s is the matrix of ∇^s (cf. (3.2)).

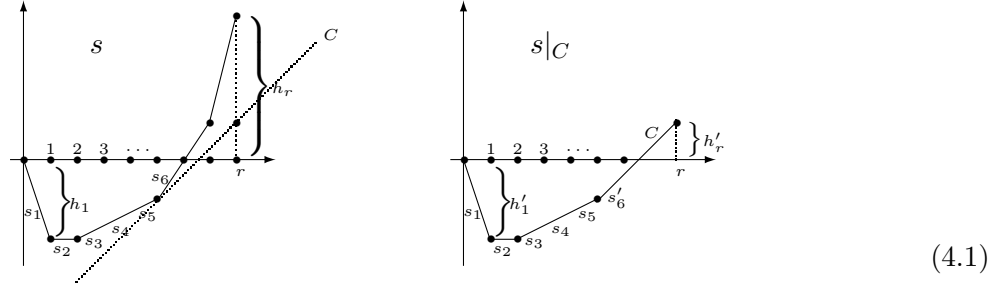
REMARK 3.8. Let $\xi \in X^{\text{gen}}$. Assume, as in Lemma 3.4, that $K = \widehat{K^{\text{alg}}}$ or, if $K \neq \widehat{K^{\text{alg}}}$, that $\xi = \xi_{c,\rho}$ with $c \in K$, $\rho > 0$. Then $\|d/dT\|_{S_p, \xi} = \omega/r(\xi)$ and $\mathcal{R}^{M, \text{sp}}(\xi) = \min(\mathcal{R}^Y(\xi), r(\xi))$.

REMARK 3.9. If $K \neq K^{\text{alg}}$ then $\mathcal{R}^{M \otimes \widehat{K^{\text{alg}}}, \text{sp}}(\xi) = \min(\mathcal{R}^Y(\xi), r(\xi))$. Both functions $\xi \mapsto r(\xi)$ and $\xi \mapsto \mathcal{R}^Y(\xi)$ are invariant by the action of $G := \text{Gal}(K^{\text{alg}}/K)$ so $\mathcal{R}^{M \otimes \widehat{K^{\text{alg}}}, \text{sp}}$ defines a function on $X = X \otimes \widehat{K^{\text{alg}}}/G$. This was the function considered in sections 3.1 and 3.2 (cf. (3.4)).

4. Newton polygons.

Let $r \geq 1$ be a natural number. Let $v : \{0, 1, \dots, r\} \rightarrow \mathbb{R} \cup \{+\infty\}$, be any sequence $i \mapsto v_i$ satisfying $v_0 = 0$. The *Newton polygon* $NP(v) \subset \mathbb{R}^2$ is the convex hull in \mathbb{R}^2 of the family of half-lines $L_v := \{(x = i, y \geq v_i)\}_{i=0, \dots, r}$ i.e. the intersection of all the upper half planes $H_{a,b} := \{(x, y) \in \mathbb{R}^2 \text{ such that } y \geq ax + b\}$, $a, b \in \mathbb{R}$, containing L_v . We call the *i-th partial height* of the polygon the value $h_i := \min\{y \in \mathbb{R} \cup \{+\infty\} \text{ such that } (i, y) \in NP(v)\}$. If $h : \{0, \dots, r\} \rightarrow \mathbb{R} \cup \{+\infty\}$ denotes the function $i \mapsto h_i$, then $NP(v) = NP(h)$, and h is the smallest function with this property. We call *slope sequence* any increasing sequence $s : \{1, \dots, r\} \rightarrow \mathbb{R} \cup \{+\infty\}$: $s_1 \leq \dots \leq s_r$. The *slope sequence of* $NP(h)$ is defined by $s_i := h_i - h_{i-1}$, $i = 1, \dots, r$, where $s_i = +\infty$ if h_i or h_{i-1} are equal to $+\infty$. The slope sequence of $NP(h)$ determines the function $h_i = s_1 + \dots + s_i$, and hence $NP(h)$. Let $s : s_1 \leq \dots \leq s_r$ be a slope sequence, the *truncated slope sequence* by the constant $C \in \mathbb{R}$ is

by definition the sequence $s|_C := (s'_i)_{i=1,\dots,r}$, where $s'_i := \min(s_i, C)$, for all i . The corresponding polygons have the following shape:



As a matter of facts in the sequel we will deal only with truncated slope sequences by a convenient constant C . Hence the i th slope s'_i , as well as the i th partial height $h'_i := s'_1 + \dots + s'_i$ will never be equal to $+\infty$. The explicit expression of h_i in terms of the v_i is $h_i = \sup_{s \in \mathbb{R}} \left(s \cdot i + \min_{j=0,\dots,r} (v_j - s \cdot j) \right)$. In fact if $y = sx + q_s$ is the line of slope s which is tangent to $NP(v)$, then $q_s = \min_{j=0,\dots,r} (v_j - s \cdot j)$, and h_i is the supremum of the values of those lines at $x = i$.

EXAMPLE 4.1. Let $(F, |\cdot|_F)$ be a valued field and let $P(T) := \sum_{i=0}^r a_{r-i} T^i \in F[T]$ be such that $a_0 = 1$. Let $v_{P,i} := -\ln(|a_i|) \in \mathbb{R} \cup \{+\infty\}$. The Newton polygon of $P(T)$ is by definition $NP(v_P)$.

4.1 Spectral Newton polygon of a differential operator.

Let $\xi \in X$. Let $\mathcal{L} := \sum_{i=0}^r g_{r-i}(T) \cdot (d/dT)^i$, with $g_0 = 1$ and $g_i \in \mathcal{O}(X)$. Let $v_{\mathcal{L}} : i \mapsto -\ln(\omega^{-i} \cdot |g_i(\xi)|)$. The Newton polygon $NP(\mathcal{L}, \xi) := NP(v_{\mathcal{L}})$ is called the *spectral Newton polygon* of \mathcal{L} . Let $s^{\mathcal{L}, \text{sp}}(\xi) : s_1^{\mathcal{L}, \text{sp}}(\xi) \leq \dots \leq s_r^{\mathcal{L}, \text{sp}}(\xi)$, be the slope sequence of $NP(\mathcal{L}, \xi)$. Then

$$s_1^{\mathcal{L}, \text{sp}}(\xi) = \ln \left(\omega \cdot \min_{i=1,\dots,r} |g_i(\xi)|^{-\frac{1}{i}} \right). \quad (4.2)$$

If $\xi \in X^{\text{gen}}$ none of the values $|g_i(\xi)|$ is equal to zero, so $s_i^{\mathcal{L}, \text{sp}}(\xi)$ and $h_i^{\mathcal{L}, \text{sp}}(\xi) := s_1^{\mathcal{L}, \text{sp}}(\xi) + \dots + s_i^{\mathcal{L}, \text{sp}}(\xi)$ are all finite. To be consistent with the rest of the paper we set $\mathcal{R}_i^{\mathcal{L}, \text{sp}}(\xi) := \exp(s_i^{\mathcal{L}, \text{sp}}(\xi))$ and $H_i^{\mathcal{L}, \text{sp}}(\xi) := \exp(h_i^{\mathcal{L}, \text{sp}}(\xi))$. Analogous definitions are given if $g_i \in \mathcal{H}(\xi)$, $\xi \in X^{\text{gen}}$.

PROPOSITION 4.2. Assume that $g_1, \dots, g_r \in \mathcal{O}(X)$ have no zeros on X , i.e. $g_i, g_i^{-1} \in \mathcal{O}(X)$ for all $i = 1, \dots, r$. Then :

- i) For all $i = 0, \dots, r$ the function $\xi \mapsto H_i^{\mathcal{L}, \text{sp}}(\xi) \in \mathbb{R}$ verifies the six properties (C1)–(C6) with respect to $\Gamma := \Gamma_X$ and $\mathcal{C}(H_i^{\mathcal{L}, \text{sp}}) := \mathcal{C}_X$, and is hence finite by Thm. 2.14. Moreover, if $K = \widehat{K^{\text{alg}}}$, then $H_i^{\mathcal{L}, \text{sp}}$ is super-harmonic in the sense of Definition 2.10;
- ii) For all $i = 0, \dots, r$ one has $\Gamma(h_i^{\mathcal{L}, \text{sp}}) = \Gamma(H_i^{\mathcal{L}, \text{sp}}) = \Gamma_X$;
- iii) Assume that $\xi \in X_{\text{int}}$, and that $(i, h_i^{\mathcal{L}, \text{sp}}(\xi))$ is a vertex of $NP(\mathcal{L}, \xi)$ (i.e. $i = r$ or $s_i^{\mathcal{L}, \text{sp}}(\xi) < s_{i+1}^{\mathcal{L}, \text{sp}}(\xi)$). Then:
 - (iii-a) There exists an open segment containing ξ of each branch in $\mathcal{B}(\xi)$ on which $H_i^{\mathcal{L}, \text{sp}} = \omega^i |g_i|^{-1}$.
 - (iii-b) In this case for all $\delta \in \Delta(\xi)$ the slopes $\partial_- H_{i,\delta}^{\mathcal{L}, \text{sp}}(\xi)$ and $\partial_+ H_{i,\delta}^{\mathcal{L}, \text{sp}}(\xi)$ are equal to those of $\xi \mapsto |g_i(\xi)|^{-1}$ and lies hence in \mathbb{Z} by the property (Z) of section 1.3.
 - (iii-c) If moreover ξ is not belonging to the Shilov boundary of X , and if $K = \widehat{K^{\text{alg}}}$, then $H_i^{\mathcal{L}, \text{sp}}$ is harmonic at ξ (i.e. (2.8) is an equality).
- iv) From (iii-b) one deduces, by interpolation, that for all $i = 1, \dots, r$ the slopes of $h_i^{\mathcal{L}, \text{sp}}$ and $s_i^{\mathcal{L}, \text{sp}}$ belong to $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{r}\mathbb{Z}$.

Proof. Since every g_i has no zeros on X the functions $\xi \mapsto |g_i(\xi)|$ are constant on every maximal disk $D^-(t, \rho_{t,X})$. Hence ii) holds. The rest is straightforward (see for example [Ked10b, Thm.11.2.1]). \square

PROPOSITION 4.3 (Small radius). *Let $\xi \in X^{\text{gen}}$, let $\mathcal{L} := \sum_{i=0}^r g_{r-i} \cdot (d/dT)^i$, $g_0 = 1$, $g_i \in \mathcal{H}(\xi)$, and let (M, ∇) be the corresponding differential module over $\mathcal{H}(\xi)$. Following [BGR84, 1.5.4] let*

$$\|\mathcal{L}\|_{Sp, \xi} := \max_{1 \leq i \leq r} |g_i(\xi)|^{\frac{1}{i}}. \quad (4.3)$$

Then $\|\mathcal{L}\|_{Sp, \xi} > |d/dT|_{\mathcal{H}(\xi)}$ if and only if $\|\nabla\|_{Sp, \xi} > |d/dT|_{\mathcal{H}(\xi)}$. In this case one has

$$\|\nabla\|_{Sp, \xi} = \|\mathcal{L}\|_{Sp, \xi}. \quad (4.4)$$

Proof. Reproduce closely the proof of [CM02, Thm.6.2]. \square

Under the assumptions of Lemma 3.4 one has $|d/dT|_{\mathcal{H}(\xi)} = 1/r(\xi)$. So Proposition 4.3 can be rephrased as follows: $\|\mathcal{L}\|_{Sp, \xi} > r(\xi)^{-1}$ if and only if $\mathcal{R}^{M, \text{sp}}(\xi) < \omega \cdot r(\xi)$, and in this case

$$\mathcal{R}^{M, \text{sp}}(\xi) = \omega / \|\mathcal{L}\|_{Sp, \xi} = \omega \cdot \min_{i=1, \dots, r} |g_i(\xi)|^{-1/i} = \exp(s_1^{\mathcal{L}, \text{sp}}(\xi)). \quad (4.5)$$

PROPOSITION 4.4. *Let $\alpha > 0$, be a constant and let $C(\xi) := \ln(\alpha \cdot r(\xi)) \in \{-\infty\} \cup \mathbb{R}$. Assume that $g_1, \dots, g_r \in \mathcal{O}(X)$ have no zeros on X . Let $s'(\xi) : s'_1(\xi) \leq \dots \leq s'_r(\xi)$ be the truncated sequence $s^{\mathcal{L}, \text{sp}}(\xi)|_{C(\xi)}$. Namely $s'_i(\xi) := \min(s_i^{\mathcal{L}, \text{sp}}(\xi), C(\xi))$ for all $i = 1, \dots, r$. In order to avoid to work with $-\infty$ let $\widehat{R'_i} := \exp(s'_i)$, and let $H'_i(\xi) := \exp(h'_i(\xi))$ where as usual $h'_i(\xi) := s'_1(\xi) + \dots + s'_i(\xi)$, and if $t \in X(K^{\text{alg}})$ we extend the above definition by $R'_i(\xi_t) := 0$ and $H'_i(\xi_t) = 0$. Then :*

- i) *For all $i = 0, \dots, r$ the function $\xi \mapsto H'_i(\xi) \in \mathbb{R}$ verifies (C2), (C4), and (C3) with $\Gamma = \Gamma_X$;*
- ii) *$\Gamma(R'_i) = \Gamma(H'_i) = X$, because $R'_i(\xi_t) = 0$ for all $t \in X(K^{\text{alg}})$;*
- iii) *Assume $\xi \in X_{\text{int}}$. If $(i, h'_i(\xi))$ is a vertex of the truncated Newton polygon at ξ (i.e. $i = r$ or $s'_i(\xi) < s'_{i+1}(\xi)$) then for all $\delta \in \Delta(\xi)$ the slopes $\partial_- H'_{i, \delta}(\xi)$ and $\partial_+ H'_{i, \delta}(\xi)$ lies in \mathbb{Z} . This implies, by interpolation, that for all $i = 1, \dots, r$ the log-slopes of H'_i always belong to $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{r}\mathbb{Z}$.*
- iv) *Assume that $\xi \in X_{\text{int}}$ does not belong to the Shilov boundary of X . Let $i_0 \in \{1, \dots, r\}$ be the largest integer such that $s'_{i_0}(\xi) < C(\xi)$. If $K = \widehat{K^{\text{alg}}}$, then for all $i = 1, \dots, i_0$, the function H'_i is super-harmonic at ξ , and if moreover $(i, h'_i(\xi))$ is a vertex of the truncated Newton polygon as in iii), then H'_i is harmonic at ξ .*

Proof. Straightforward [Ked10b, Remark 11.2.4]. \square

4.2 Spectral Newton polygon of a differential module.

By Lemma 3.4 if $\xi \in X^{\text{gen}}$, then d/dT extends by continuity to $\mathcal{H}(\xi)$. Let $M = M_\xi$ be a differential module of rank r over $(\mathcal{H}(\xi), d/dT)$. Let $0 = M_{\xi, 0} \subset M_{\xi, 1} \subset \dots \subset M_{\xi, n} = M$ be a Jordan-Hölder sequence of M . This means that $N_{\xi, k} := M_{\xi, k} / M_{\xi, k-1}$ has no non trivial strict differential submodules for all k . Let r_k be the rank of $N_{\xi, k}$, and let $R_k := \mathcal{R}^{N_{\xi, k}, \text{sp}}(\xi)$. Perform a permutation of the indexes in order to have $R_1 \leq \dots \leq R_n$. Let $s^{M, \text{sp}}(\xi) : s_1^{M, \text{sp}}(\xi) \leq \dots \leq s_r^{M, \text{sp}}(\xi)$ be the slope sequence obtained from $\ln(R_1) \leq \dots \leq \ln(R_n)$ by counting r_k -times the slope $\ln(R_k)$:

$$s^{M, \text{sp}}(\xi) : \underbrace{\ln(R_1) = \dots = \ln(R_1)}_{r_1\text{-times}} \leq \underbrace{\ln(R_2) = \dots = \ln(R_2)}_{r_2\text{-times}} \leq \dots \leq \underbrace{\ln(R_n) = \dots = \ln(R_n)}_{r_n\text{-times}}. \quad (4.6)$$

Set $h_0^{M, \text{sp}}(\xi) = 0$ and $h_i^{M, \text{sp}}(\xi) := s_1^{M, \text{sp}}(\xi) + \dots + s_i^{M, \text{sp}}(\xi)$, for all $i = 1, \dots, r$. The *spectral Newton polygon* $NP^{\text{sp}}(M, \xi)$ is by definition $NP(h^{M, \text{sp}}(\xi))$. We also set $\mathcal{R}_i^{M, \text{sp}}(\xi) := \exp(s_i^{M, \text{sp}}(\xi))$, and $H_i^{M, \text{sp}}(\xi) := \exp(h_i^{M, \text{sp}}(\xi))$. One has $\mathcal{R}_1^{M, \text{sp}}(\xi) = \mathcal{R}^{M, \text{sp}}(\xi)$. As for $\mathcal{R}^{M, \text{sp}}(\xi)$ (cf. Def. 3.6), we extend the definition of $\mathcal{R}_i^{M, \text{sp}}(\xi)$ to the whole X by setting $\mathcal{R}_i^{M, \text{sp}}(\xi) = 0$, for all $\xi \in X - X^{\text{gen}}$.

4.3 Convergence Newton polygon of a differential module and main theorem

Let M be a differential module over $\mathcal{O}(X)$ of rank r . Let $\xi \in X$ and let $t \in X(\Omega)$ be a Dwork generic point for ξ . For all $0 < R \leq \rho_{\xi, X}$ denote by $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) \subset M \otimes \mathcal{A}_\Omega(t, R)$ the Ω -vector space of solutions of M with values in $\mathcal{A}_\Omega(t, R)$. If $R \leq \mathcal{R}^M(\xi)$, then $\text{Sol}(M, t, \Omega) := \text{Fil}^{\geq R} \text{Sol}(M, t, \Omega)$ is independent on $R \leq \mathcal{R}^M(\xi)$, and it has dimension r over Ω . The family $\{\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega)\}_{0 < R \leq \rho_{\xi, X}}$ is a descending filtration of $\text{Sol}(M, t, \Omega)$ by Ω -sub-vector spaces.

The filtration is independent on the choice of Ω and t in the following sense. A descent argument shows that if $t \in \Omega \subset \Omega'$, then $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega') = \text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) \otimes_\Omega \Omega'$ (cf. [Ked10b, Prop. 6.9.1]). If now $t' \in \Omega$ is another Dwork generic point for ξ , then, up to enlarge Ω , one has $t' = \sigma(t)$ for some $\sigma \in \text{Gal}^{\text{cont}}(\Omega/K)$ (cf. Lemma 1.2). Since σ is isometric, then $\sigma(D^-(t, R)) = D^-(t', R)$ for all $0 < R \leq \rho_{\xi, X}$. This provides an isomorphism of rings $\sum a_i(T-t)^i \mapsto \sum \sigma(a_i)(T-t')^i : \mathcal{A}_\Omega(t, R) \xrightarrow{\sim} \mathcal{A}_\Omega(t', R)$ commuting with d/dT . Hence $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega)$ is identified by σ to $\text{Fil}^{\geq R} \text{Sol}(M, t', \Omega)$.

For all $i = 1, \dots, r$ define $\mathcal{R}_i^M(\xi)$ as the largest value of $R \leq \rho_{\xi, X}$ such that $\dim_\Omega \text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) \geq r - i + 1$. In other words $\mathcal{R}_i^M(\xi)$ is the radius of the largest open disk centered at t on which M has at least $r - i + 1$ linearly independent solutions. For all $i = 1, \dots, r$ set $s_i^M(\xi) := \ln(\mathcal{R}_i^M(\xi))$ and $h_i^M(\xi) := s_1^M(\xi) + \dots + s_i^M(\xi)$, $h_0^M(\xi) = 0$. By the above arguments $s_i^M(\xi)$ and $h_i^M(\xi)$ only depend on M and ξ , and are independent on the choice of t and Ω . The *convergence Newton polygon* $NP^{\text{conv}}(M, \xi)$ is by definition $NP(h^M(\xi))$. We also set $H_i^M(\xi) := \exp(h_i^M(\xi))$. One has (cf. (3.3))

$$\mathcal{R}_1^M(\xi) = \mathcal{R}^M(\xi). \quad (4.7)$$

For all $t, t' \in X(\Omega)$ satisfying $|t - t'| < R \leq \rho_{\xi, X}$ one has a canonical identification

$$\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) \xrightarrow{\sim} \text{Fil}^{\geq R} \text{Sol}(M, t', \Omega) \quad (4.8)$$

induced by the canonical isomorphism $f(T) \mapsto \sum_{n \geq 0} f^{(n)}(t')(T - t')^n/n! : \mathcal{A}_\Omega(t, R) \xrightarrow{\sim} \mathcal{A}_\Omega(t', R)$ which commutes with d/dT . So a solution in $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega)$ converges at all point $t' \in D^-(t, R)$ and belongs to $\text{Fil}^{\geq R} \text{Sol}(M, t', \Omega)$. This proves that for all $i = 1, \dots, r$ one has

$$\max(\mathcal{R}_i^M(\xi), r(\xi)) \leq \rho_{\mathcal{R}^M(\xi)}. \quad (4.9)$$

DEFINITION 4.5. If M is a differential module over $\mathcal{A}_\Omega(t, \rho)$, and if $\xi \in D^-(t, \rho)$, one defines $\mathcal{R}_i^M(\xi)$, $\text{Fil}^{\geq R} \text{Sol}(M, t_\xi, \Omega)$, $s_i^M(\xi)$, $h_i^M(\xi)$, $NP^{\text{conv}}(M, \xi)$, replacing everywhere $\rho_{\xi, X}$ by ρ .

REMARK 4.6. By construction \mathcal{R}_i^M (and hence s_i^M , h_i^M , H_i^M) are invariant by scalar extension of the field K . The spectral radius $\mathcal{R}^{M, \text{sp}}$ does not enjoy this property. So, along a branch $\Lambda(t)$, with $t \in X(\Omega)$, we will be obliged to compare $\mathcal{R}_{i, t}^M(\rho)$ with $\mathcal{R}_{i, t}^{M \otimes \Omega, \text{sp}}(\rho)$ along $[0, R_0]$.

The main result of this paper is the following:

THEOREM 4.7. Let M be a differential module of rank r over $\mathcal{O}(X)$. For all $i = 1, \dots, r$ the functions \mathcal{R}_i^M (and hence s_i^M , h_i^M , H_i^M) are all finite. They enjoy moreover the following properties:

- i) $[\mathcal{R}_i^M \text{ VS } \mathcal{R}_i^{M, \text{sp}}]$ If $t \in X(\Omega)$ and $\rho \in [0, R_0]$, then for all $i = 1, \dots, r$ one has

$$\mathcal{R}_{i, t}^{M \otimes \Omega, \text{sp}}(\rho) = \min(\mathcal{R}_{i, t}^M(\rho), \rho) \quad \text{and} \quad \mathcal{R}_{i, t}^M(\rho) = \begin{cases} \mathcal{R}_i^M(t) & \text{if } \rho \in [0, \mathcal{R}_i^M(t)] \\ \mathcal{R}_{i, t}^{M \otimes \Omega, \text{sp}}(\rho) & \text{if } \rho \in [\mathcal{R}_i^M(t), R_0] \end{cases} \quad (4.10)$$

as functions on $[0, R_0]$ (cf. Remark 4.6). In particular $\mathcal{R}_i^{M \otimes \widehat{K}^{\text{alg}}, \text{sp}} = \mathcal{R}_i^M$ along Γ_X .

- ii) *[Integrality]* Assume $\xi \in X_{\text{int}}$. If $(i, h_i^M(\xi))$ is a vertex of $NP^{\text{conv}}(M, \xi)$ (i.e. $i = r$ or $s_i^M(\xi) < s_{i+1}^M(\xi)$), then for all $\delta \in \Delta(\xi)$, the log-slopes $\partial_- H_{i, \delta}^M(\xi)$ and $\partial_+ H_{i, \delta}^M(\xi)$ lies in \mathbb{Z} . This implies, by interpolation, that for all $i = 1, \dots, r$ the log-slopes $\partial_- H_{i, \delta}^M(\xi)$ and $\partial_+ H_{i, \delta}^M(\xi)$ always belong to $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{r}\mathbb{Z}$.

- iii) For all $i = 1, \dots, r$ the function $H_i^M : X \rightarrow \mathbb{R}_{>0}$ satisfies (C1), (C2), (C4), (C5). Let $\Gamma_0 := \Gamma_X$, and $\Gamma_i := \bigcup_{j=1}^i \Gamma(\mathcal{R}_j^M)$. Then H_i^M satisfies (C3) with respect to $\Gamma := \Gamma_{i-1}$. More precisely H_i^M / \mathcal{R}_i^M is constant on each (non generic) disk tangent to Γ_{i-1} , and \mathcal{R}_i^M enjoys on that disk all the properties of a genuine radius of convergence (cf. Prop. 7.5). In particular if $\xi \notin \Gamma_{i-1}$, then $\xi \in \Gamma(\mathcal{R}_i^M)$ if and only if $\partial_- \mathcal{R}_i^M(\xi) < 0$.
- iv) [Concavity] Let $t \in X(\Omega)$, $\Omega \in E(K)$, and let $I \subseteq [0, R_0]$ be a subinterval with interior $\overset{\circ}{I}$. Then
- (a) If $\rho_{t,X} \leq \inf(I)$, and if the open annulus $\{|T - t| \in \overset{\circ}{I}\}$ is contained in $X \hat{\otimes} \Omega$, then ${}^L H_{i,t}^M$ is concave on $\ln(I)$, and equal to ${}^L H_{i,t}^{M \hat{\otimes} \Omega, \text{sp}}$ on it.
 - (b) Assume $\sup(I) \leq \rho_{t,X}$. Then :
 - i. $H_{i,t}^M$ is log-concave on each open subinterval $J \subseteq \overset{\circ}{I}$ satisfying $J \cap Q_t = \emptyset$, where $Q_t := \{\ln(\mathcal{R}_2^M(t)), \ln(\mathcal{R}_3^M(t)), \dots, \ln(\mathcal{R}_i^M(t))\}$. More generally if $\rho \in Q_t$ and if for all j such that $\rho = \mathcal{R}_j^M(t)$ the function $\rho \mapsto {}^L \mathcal{R}_{j,t}^M(\rho)$ is concave at ρ (i.e. $\partial_+ \mathcal{R}_j^M(\xi_{t,\rho}) < 0$), then one can replace the set Q_t by $Q_t - \{\rho\}$.
 - ii. Let $\rho \in]0, \rho_{t,X}[$. If for all $k = 1, \dots, i$ one has $\mathcal{R}_k^M(\xi_{t,\rho}) \neq \rho$, then the left and right log-slopes of $H_{i,t}^M$ at ρ are less than or equal to 0 (i.e. $H_{i,t}^M$ is log-decreasing at ρ).
- v) [Weak super-harmonicity] Assume $K = \widehat{K^{\text{alg}}}$. Let \mathcal{C}_1 be the Shilov boundary of X , and, inductively, define $\mathcal{C}_i := \mathbb{N}_i \cup (\bigcup_{j=0}^{i-1} \mathcal{C}_j)$, where \mathbb{N}_i is the finite set of points $\xi \in X$ satisfying (a) $\xi \in \Gamma(\mathcal{R}_i^M) \cap \Gamma(H_i^M) \cap \Gamma_{i-1}$; (b) $\mathcal{R}_i^M(\xi) = r(\xi)$; (c) ξ lies in the boundary of $\Gamma(\mathcal{R}_i^M)$. Then for all $i = 1, \dots, r$ the function H_i^M is super-harmonic (at least) at all $\xi \in X_{\text{int}} - \mathcal{C}_i$.
- vi) [Weak harmonicity of the vertexes] Assume $K = \widehat{K^{\text{alg}}}$ and let $\xi \in X_{\text{int}}$ be not in the Shilov boundary of X . If $\xi \notin \Gamma(H_i^M)$, then H_i^M is flat and hence harmonic at ξ . Assume then $\xi \in \Gamma(H_i^M)$. In this case if none of the values $\{\mathcal{R}_j^M(\xi)\}_{j=1, \dots, i}$ is equal to $r(\xi)$, and if $(i, h_i^M(\xi))$ is a vertex of $NP^{\text{conv}}(M, \xi)$ (i.e. $i = r$, or $s_i^M(\xi) < s_{i+1}^M(\xi)$), then H_i^M is harmonic at ξ .

COROLLARY 4.8. Let $t \in K$, and $A(t, I) := \{|T - t| \in I\}$ be a possibly not closed annulus or disk (if $0 \in I$ one has a disk). Let \mathcal{O} be one of the rings $\mathcal{H}_K(t, I)$, $\mathcal{B}_K(t, I)$, $\mathcal{A}_K(t, I)$. (cf. Section 1.0.3). Let M be a differential module over \mathcal{O} of rank r . Then Thm. 4.7 holds for M in the following cases¹¹

- i) if $\mathcal{O} = \mathcal{H}_K(t, I)$;
- ii) if K is discretely valued, and $\mathcal{O} = \mathcal{B}_K(t, I)$;
- iii) if $\mathcal{O} = \mathcal{B}_K(t, I)$ or $\mathcal{O} = \mathcal{A}_K(t, I)$, and all $\mathcal{R}_1^M, \dots, \mathcal{R}_r^M$ (or equivalently all H_i^M) have a finite number of breaks along $\{\xi_{t,\rho}\}_{\rho \in I}$.¹²

Moreover if $\mathcal{O} = \mathcal{B}_K(t, I)$ or $\mathcal{O} = \mathcal{A}_K(t, I)$, and if there exists $i \leq r$ such that all $\mathcal{R}_1^M, \dots, \mathcal{R}_i^M$ have a finite number of breaks along $\{\xi_{t,\rho}\}_{\rho \in I}$, then $\mathcal{R}_1^M, \dots, \mathcal{R}_i^M$ (and hence H_1^M, \dots, H_i^M) are finite.

The proof of Corollary 4.8 is placed at the end of section 7.

REMARK 4.9. By v) of Thm. 4.7 one has the following statements.

- i) If $K = \widehat{K^{\text{alg}}}$, then the function \mathcal{R}_1^M is super-harmonic on X (cf. Def. 2.10).
- ii) If none of the values $\{\mathcal{R}_j^M(\xi)\}_{j=1, \dots, i}$ is equal to $r(\xi)$, then $\partial_- H_{i,\delta}^M(\xi) = 0$ for almost but a finite number of directions $\delta \in \Delta(\xi)$, and H_i^M is super-harmonic at ξ .

¹¹See Definition 2.20 for the notion of finiteness over a possibly not open ring or annulus.

¹²If $A(t, I)$ is a disk $D^-(t, \rho)$, and if $\lim_{\rho' \rightarrow \rho^-} \mathcal{R}_{i,t}^M(\rho') = \rho$, then the same holds for all $j \geq i$, and in this case \mathcal{R}_j^M has a finite number of breaks along $\{\xi_{t,\rho}\}_{\rho \in I}$ (cf. proof of Cor. 4.8).

- iii) Let $D^-(t, \rho)$ be a non generic disc. If $D^-(t, \rho) \cap \Gamma_{i-1} = \emptyset$, then \mathcal{R}_i^M and H_i^M differs by a constant on $D^-(t, \rho)$, and $\Gamma(\mathcal{R}_i^M) \cap D^-(t, \rho) = \Gamma(H_i^M) \cap D^-(t, \rho)$. \mathcal{R}_i^M and H_i^M enjoy the six properties on (C1)–(C5) with respect to $\Gamma := \Gamma_{i-1}$ on that disk, in particular they are concave and decreasing on each branch $\Lambda(\xi')$ for all $\xi' \in D^-(t, \rho)$. If moreover $K = \widehat{K^{\text{alg}}}$, then they are both super-harmonic at all $\xi \in D^-(t, \rho)$.
- iv) If M has rank $r = 1$, then the boundary points of $\Gamma(\mathcal{R}^M)$ not in the Shilov boundary of X are those $\xi \in X - \Gamma_X$ satisfying $\mathcal{R}^M(\xi) = r(\xi)$ and $\partial_+ \mathcal{R}^M(\xi) < 0$ (cf. Cor. 7.9).
- v) Assume K discretely valued. Since $|a| = 1$ for all $a \neq 0$ it follows from section 1.0.3 that $K((T)) = \mathcal{A}_K(0, I) = \mathcal{B}_K(0, I)$, for all (open or closed) interval $I \subseteq]0, 1[$. Moreover $K[T, T^{-1}] = \mathcal{A}_K(0, I) = \mathcal{B}_K(0, I)$ for all I such that $1 \in I$ and $0 \notin I$. One has analogous interpretations for $K[[T]] = \mathcal{A}_K(0, R) = \mathcal{B}_K(0, [0, R[)$, if $R \leq 1$, and $K[T] = \mathcal{A}_K(0, R) = \mathcal{B}_K(0, [0, R[)$, if $R > 1$. Corollary 4.8 holds if M is a differential module over $K((T))$, or $K[T, T^{-1}]$. In this case $\omega = 1$, and the slopes $\{\partial_- s_{i,0}^M\}_i$ are directly related to the Formal Newton polygon of M [Ram78], [DMR07, p.97–107], [Rob80].

The remaining of the paper is devote to prove Theorem 4.7. For expository reasons below we give a first part of the proof (cf. section 4.3.1).

REMARK 4.10. (C1) follows immediately by (4.9). One has $\mathcal{R}_1^{M \widehat{\otimes} K^{\text{alg}}, \text{sp}} = \mathcal{R}_1^M$ along the branches of $\Gamma(\mathcal{R}_1^M)$. In fact if $\xi \in \Gamma(\mathcal{R}^M)$ then $r(\xi) = \rho_{\mathcal{R}^M(\xi)}$ and one has (3.9).

4.3.1 *Comparison between \mathcal{R}_i^M and $\mathcal{R}_i^{M, \text{sp}}$.* We now prove point i) of Theorem 4.7. Firstly we prove, for all $\rho \in [0, R_0]$, the equality (4.10) :

$$\mathcal{R}_{i,t}^{M \widehat{\otimes} \Omega, \text{sp}}(\rho) = \min(\mathcal{R}_i^M(\rho), \rho). \quad (4.11)$$

For $\rho = 0$ both functions are 0 and there is nothing to prove. Assume $\rho > 0$. Since \mathcal{R}_i^M is insensitive by scalar extension of K we can assume $\Omega = K$, $t \in X(K)$. Let $t_\rho \in X(\Omega_\rho)$ be a Dwork generic point for $\xi_{t,\rho}$ and let $D^-(t_\rho, \rho) \subset X \widehat{\otimes} \Omega_\rho$ be the generic disk with $\rho = r(\xi_{t,\rho})$. It is a classical idea of Dwork that the restriction map $\mathcal{O}(X) \rightarrow \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)$ sending f into $f|_{D^-(t_\rho, \rho)}(T) = \sum_{n \geq 0} f^{(n)}(t_\rho)(T - t_\rho)^n / n!$ factorizes through an injective map $\mathcal{O}(X) \rightarrow \mathcal{H}(\xi_{t,\rho}) \subset \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)$. The truncated sequence $\{s_i^M(\xi_{t,\rho})|_{\ln(\rho)} := \min(s_i^M(\xi_{t,\rho}), \ln(\rho))\}_i$ coincides with the slope sequence $\{s_i^{M \otimes_{\mathcal{O}(X)} \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)}(\xi_{t,\rho})\}_i$ of the restricted module $M \otimes_{\mathcal{O}(X)} \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)$ (cf. Def. 4.5). Both $\{s_i^{M, \text{sp}}(\xi_{t,\rho})\}_i$ and $\{s_i^{M \otimes_{\mathcal{O}(X)} \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)}(\xi_{t,\rho})\}_i$ only depends on $M_{\xi_{t,\rho}} := M \otimes_{\mathcal{O}(X)} \mathcal{H}(\xi_{t,\rho})$, so we can assume $M = M_{\xi_{t,\rho}}$. By Thm. 5.1 below we can assume $M_{\xi_{t,\rho}} = M_{\xi_{t,\rho}}^R$. By definition of $M_{\xi_{t,\rho}}^R$ one has $\mathcal{R}_i^{M_{\xi_{t,\rho}}^R, \text{sp}}(\xi_{t,\rho}) = R$ for all $i = 1, \dots, r$, and by (3.12) and (3.16) one has $R = \mathcal{R}^{M_{\xi_{t,\rho}}^R}(\xi_{t,\rho})$. By [Rob75] (cf. also [CM02], [Chr12]) $M_{\xi_{t,\rho}}^R$ also admits a decomposition by the radii of convergence of its solutions around t_ρ . So by (3.12) for all $i = 2, \dots, r$ one has $\mathcal{R}_i^{M_{\xi_{t,\rho}}^R}(\xi_{t,\rho}) = \mathcal{R}_1^{M_{\xi_{t,\rho}}^R}(\xi_{t,\rho})$. Alternatively for all $r \leq \rho$ let $N := M_{\xi_{t,\rho}}^R \otimes_{\mathcal{H}(\xi_{t,\rho})} \mathcal{A}_{\Omega_\rho}(t_\rho, \rho)$. By Remark 3.8 for all i one has $\mathcal{R}_{i,t_\rho}^{N, \text{sp}}(\rho') = \min(R, \rho')$ for all $\rho' \leq \rho$. By contrapositive if there exists r such that $R < r < \mathcal{R}_i^{M_{\xi_{t,\rho}}^R}(\xi_{t,\rho})$, then $N \otimes_{\mathcal{A}_{\Omega_\rho}(t_\rho, \rho)} \mathcal{H}(\xi_{t,\rho,r})$ would have a trivial submodule in its Jordan-Hölder sequence, and hence $\mathcal{R}_{i,t_\rho}^{N, \text{sp}}(r) \geq r > R$ for some i which is absurd. This is also an old idea of Dwork [Dwo73] and Robba [Rob75]: the decomposition by the spectral radius is the decomposition by the radius at t_ρ , see also [CD94], [CM02], [Chr12]. The proof presented here comes from [Ked10b, Thm. 11.9.2].

LEMMA 4.11. Equality $\mathcal{R}_{i,t}^M(\rho) = \begin{cases} \mathcal{R}_i^M(t) & \text{if } \rho \in [0, \mathcal{R}_i^M(t)] \\ \mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho) & \text{if } \rho \in [\mathcal{R}_i^M(t), R_0] \end{cases}$ follows from (4.11). Moreover the function $\rho \mapsto \mathcal{R}_{i,t}^M(\rho)$ is continuous on $[0, R_0]$.

Proof. By (4.9), \mathcal{R}_i^M is constant on the disk $D^-(t, \mathcal{R}_i^M(t))$ with value $R := \mathcal{R}_i^M(t)$. So $\mathcal{R}_{i,t}^M(\rho) = R$, for all $\rho \in [0, R[$. Again by (4.9), for all $\rho \in [\mathcal{R}_i^M(t), \rho_{t,X}[$ one has $\mathcal{R}_{i,t}^M(\rho) \leq \rho$. So, by (4.11), one has $\mathcal{R}_{i,t}^M(\rho) = \mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho)$ for all $\rho \in [R, \rho_{t,X}[$. If $\rho \in [\rho_{t,X}, R_0]$, then $\rho = \rho_{\xi_{t,\rho}, X}$ and $\mathcal{R}_{i,t}^M(\rho) \leq \rho$ by definition, and is hence equal to $\mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho)$ by (4.11). Now by Thm. 5.6 below, $\rho \mapsto \mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho)$ is continuous on $[0, R_0]$. Hence $\mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(R) = R$ and $\rho \mapsto \mathcal{R}_{i,t}^M(\rho)$ is continuous too. In fact otherwise the condition $\mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(R) < R$ implies $\mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho) < \rho$ in a neighborhood of $\rho = R$. Hence $\mathcal{R}_{i,t}^{M\hat{\otimes}\Omega, \text{sp}}(\rho) < \rho < R$ for $\rho < R$, contradicting (4.11). \square

5. Auxiliary results

5.1 Decomposition theorems

THEOREM 5.1 (Decomposition over a point by the slopes of $NP^{\text{sp}}(M, \xi)$). Let $\xi \in X_{\text{int}}$, and let $\mathcal{L} = \sum_{i=0}^r g_{r-i}(T) \cdot (d/dT)^i$, $g_0 = 1$, $g_i \in \mathcal{H}(\xi)$ be a differential operator defining the module M_ξ . Assume as in Lemma 3.4 that $K = \widehat{K^{\text{alg}}}$ or, if $K \neq \widehat{K^{\text{alg}}}$, that $\xi = \xi_{c,\rho}$, with $c \in K$ and $\rho > 0$. Then

- i) M_ξ decomposes into a direct sum $M_\xi = \bigoplus_{0 < R \leq r(\xi)} M_\xi^R$ where $\mathcal{R}^{M, \text{sp}}(M_\xi^R) = R$ and the spectral Newton polygon $NP^{\text{sp}}(M_\xi^R, \xi)$ has a constant slope sequence of value $s_i^{M_\xi^R}(\xi) = \ln(R)$, $\forall i$.
- ii) Let $C_\omega := \ln(\omega \cdot r(\xi)) = \ln(\omega / |d/dT|_{\mathcal{H}(\xi)})$. Let $s^{\mathcal{L}, \text{sp}}(\xi)$ (resp. $s^{M, \text{sp}}(\xi)$) be the slope sequence of \mathcal{L} (resp. M). Then $s^{M, \text{sp}}(\xi)|_{C_\omega} = s^{\mathcal{L}, \text{sp}}(\xi)|_{C_\omega}$.

Proof. The original proof of i) is due Dwork [Dwo73] and Robba [Rob75], [Rob80], in the case $\xi = \xi_{0,1}$. The generalization to a point of type $\xi_{0,\rho}$ with $\rho > 0$, can be found in [CM02]. One finds in [Ked10b, Thm.6.6.1 and 10.6.2] a proof which is closer to our context. Point ii) is [You92]. Translating by an element of K the theorem holds for a point $\xi \in X_{\text{int}}$ satisfying the assumptions. \square

REMARK 5.2. 1. The first assertion of Theorem 5.1 without the second one would be quite empty. In fact the dimensions of the terms of the Jordan Hölder sequence of Definition 4.2 are unknown.

2. The first part of the proof (cf. [Ked10b, Thm.6.6.1]) holds for all points $\xi \in X^{\text{gen}}$, the second part of the proof [Ked10b, Thm. 10.6.2] requires Frobenius techniques and is stated only for points of type (2) or (3) i.e. in X_{int} . The result probably holds for all $\xi \in X^{\text{gen}}$.

COROLLARY 5.3. Assume $K = \widehat{K^{\text{alg}}}$, and let $\xi \in X_{\text{int}}$. Let M be a differential module defined by an operator $\mathcal{L} = \sum_{i=0}^r g_{r-i}(T)(d/dT)^i$, $g_0 = 1$, with $g_i, g_i^{-1} \in \mathcal{O}(X)$. If $\mathcal{R}_{i_0}^{M, \text{sp}}(\xi) < \omega \cdot r(\xi)$, then for all $1 \leq i \leq i_0$ and for all direction $\delta \in \Delta(\xi)$ one has

$$\partial_- H_{i,\delta}^{M, \text{sp}}(\xi) = \partial_- H_{i,\delta}^{\mathcal{L}, \text{sp}}(\xi), \quad \partial_+ H_i^{M, \text{sp}}(\xi) = \partial_+ H_i^{\mathcal{L}, \text{sp}}(\xi). \quad (5.1)$$

Proof. Thm. 5.1 holds for all ξ' in a small open segment containing ξ of each branch through ξ . By Lemma 4.11 $\mathcal{R}_{i_0}^{M, \text{sp}}$ are continuous so the assumption $\mathcal{R}_{i_0}^{M, \text{sp}}(\xi) < \omega \cdot r(\xi)$ holds for ξ' close to ξ . \square

THEOREM 5.4 ([Ked10b, 12.4.1]). Let $t \in K$, $\rho > 0$. Let M be a differential module over $\mathcal{A}_K(t, \rho)$ of rank r . Assume that for some $i_0 \in \{1, \dots, r-1\}$ there exists $\varepsilon > 0$ such that for all $\rho' \in]\rho - \varepsilon, \rho[$ $h_{i_0,t}^{M, \text{sp}}(\rho')$ is constant and $s_{i_0,t}^{M, \text{sp}}(\rho') < s_{i_0+1,t}^{M, \text{sp}}(\rho')$. Then $M = M_1 \oplus M_2$, where:

- i) M_1 has rank i_0 and for all $i = 1, \dots, i_0$ one has $s_{i,t}^{M_1, \text{sp}}(\rho') = s_{i,t}^{M, \text{sp}}(\rho')$ for all $\rho' \in]\rho - \varepsilon, \rho[$.
- ii) M_2 has rank $r - i_0$ and for all $i = i_0 + 1, \dots, r$ one has $s_{i-i_0,t}^{M_2, \text{sp}}(\rho') = s_{i,t}^{M, \text{sp}}(\rho')$ for all $\rho' \in]\rho - \varepsilon, \rho[$. \square

PROPOSITION 5.5. Let $M = M_1 \oplus M_2$ be a direct sum of differential modules over $\mathcal{O}(X)$ (resp. $\mathcal{A}_K(t, \rho)$) of ranks r_1 and r_2 . Then for all $\xi \in X$ (resp. $\xi \in D^-(t, \rho)$) one has up to permutation¹³

$$\{\mathcal{R}_1^M(\xi), \dots, \mathcal{R}_{r_1+r_2}^M(\xi)\} = \{\mathcal{R}_1^{M_1}(\xi), \dots, \mathcal{R}_{r_1}^{M_1}(\xi)\} \cup \{\mathcal{R}_1^{M_2}(\xi), \dots, \mathcal{R}_{r_2}^{M_2}(\xi)\}. \quad (5.2)$$

Proof. $Y_M = \begin{pmatrix} Y_{M_1} & 0 \\ 0 & Y_{M_2} \end{pmatrix}$, and $\text{Sol}(M, t, \Omega) = \text{Sol}(M_1, t, \Omega) \oplus \text{Sol}(M_2, t, \Omega)$. More precisely if $\vec{y} \in \text{Sol}(M, t, \Omega)$, the first r_1 (resp. the last r_2) entries of \vec{y} forms a solution $\vec{y}_1 \in \text{Sol}(M_1, t, \Omega)$ (resp. $\vec{y}_2 \in \text{Sol}(M_2, t, \Omega)$). This proves that $\mathcal{R}(\vec{y}, t) = \min(\mathcal{R}(\vec{y}_1, t), \mathcal{R}(\vec{y}_2, t))$ because by definition the radius of \vec{y} is the minimum of the radii of its entries. \square

5.2 Behavior of the spectral Newton polygon along a branch

THEOREM 5.6 ([Ked10b, Thm.11.3.2]). For simplicity assume $K = \widehat{K^{\text{alg}}}$. Let M be a differential module over $\mathcal{O}(X)$. Let $t \in K$, $|t - c_0| < R_0$ be a K -rational point of $D^-(c_0, R_0)$ (cf. (1.2)). Then :

- i) The functions $\rho \mapsto \mathcal{R}_{i,t}^{M,\text{sp}}(\rho)$ and $\rho \mapsto H_{i,t}^{M,\text{sp}}(\rho)$ verify properties (C2) and (C4) of section 2.3.1 along I_t (cf. notation (1.3) and (2.5)). If moreover $I \subset I_t$ is an interval with interior $\overset{\circ}{I}$ such that the annulus $\{|T - t| \in \overset{\circ}{I}\}$ is contained in X , then $\rho \mapsto H_{i,t}^{M,\text{sp}}(\rho)$ is log-concave on I .
- ii) Let $\xi \in X_{\text{int}}$. If $(i, h_i^{M,\text{sp}}(\xi))$ is a vertex of $NP^{\text{sp}}(M, \xi)$ (i.e. $i = r$ or $s_i^{M,\text{sp}}(\xi) < s_{i+1}^{M,\text{sp}}(\xi)$), then for all $\delta \in \Delta(\xi)$ one has $\partial_- H_{i,\delta}^{M,\text{sp}}(\xi), \partial_+ H_{i,\delta}^{M,\text{sp}}(\xi) \in \mathbb{Z}$. This implies, by interpolation, that for all $i = 1, \dots, r$ the log-slopes $\partial_- H_{i,\delta}^{M,\text{sp}}(\xi)$ and $\partial_+ H_{i,\delta}^{M,\text{sp}}(\xi)$ always belong to $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{r}\mathbb{Z}$.
- iii) Let $t \in X(\Omega)$ and let $\xi = \xi_{t,\bar{\rho}} \in X_{\text{int}}$ be a point such that there exists an open annulus $\{|T - t| \in [\bar{\rho} - \varepsilon, \bar{\rho} + \varepsilon]\}$, with $\varepsilon > 0$, contained in X . Let $i_0 \in \{1, \dots, r\}$ be the largest integer such that $s_{i_0}^{M,\text{sp}}(\xi_{t,\bar{\rho}}) < \ln(\bar{\rho}) = \ln(r(\xi_{t,\bar{\rho}}))$. Then for all $i = 1, \dots, i_0$, the function $H_i^{M,\text{sp}}$ verifies $\partial_- H_{i,\delta}^{M,\text{sp}}(\xi_{t,\bar{\rho}}) = 0$ for almost but a finite number of directions $\delta \in \Delta(\xi_{t,\bar{\rho}})$, and it is super-harmonic at ξ .¹⁴ Moreover if $i \leq i_0$, and if $(i, h_i^{M,\text{sp}}(\xi))$ is a vertex of the spectral Newton polygon (i.e. $i = r = i_0$, or $s_i^{M,\text{sp}}(\xi_{t,\bar{\rho}}) < s_{i+1}^{M,\text{sp}}(\xi_{t,\bar{\rho}})$), then $H_i^{M,\text{sp}}$ is harmonic at ξ .
- iv) If $\bar{\rho} < \rho_{t,X}$ and if $\mathcal{R}_{i,t}^{M,\text{sp}}(\bar{\rho}) < \bar{\rho}$, then the left and right log-slopes of $\rho \mapsto H_{i,t}^{M,\text{sp}}(\rho)$ at $\bar{\rho}$ are less than or equal to 0 (the function $\rho \mapsto H_{i,t}^{M,\text{sp}}(\rho)$ is logarithmically non increasing at $\bar{\rho}$).

Proof. The assumptions guarantee that one can assume $\Omega = K$, and $t = 0$ by a translation. In [Ked10b, Thm.11.3.2] these facts are proved for an annulus.¹⁵ The general statement is easily deduced as follows. Let $0 < \rho_1 < \rho_2 < \dots < \rho_m < R_0$ be the values of ρ for which there exists a center of a hole c_i of X with valuation $|c_i| = \rho$. Since the holes are finite in number and since the spectral Newton polygon of M only depend on its restriction $M_{\xi_{t,\rho}}$ to $\mathcal{H}(\xi_{t,\rho})$ (cf. section 1.0.3), we can assume that X equals an annulus $\{|T| \in [\rho_i, \rho_{i+1}]\}$ having possibly some holes placed at distance ρ_i and ρ_{i+1} . In this situation let $\mathcal{O}_{\text{an}}([\rho_i, \rho_{i+1}])$ be the ring of analytic elements on the open annulus $\{|T| \in [\rho_i, \rho_{i+1}]\}$. For all $\rho \in [\rho_i, \rho_{i+1}]$ the morphism $\mathcal{O}(X) \rightarrow \mathcal{H}(\xi_{t,\rho})$ factorizes through the natural restriction map $\mathcal{O}(X) \rightarrow \mathcal{O}_{\text{an}}([\rho_i, \rho_{i+1}]) \rightarrow \mathcal{H}(\xi_{t,\rho})$. So we can assume M to be a differential module over $\mathcal{O}_{\text{an}}([\rho_i, \rho_{i+1}])$. The assertions are hence proved in [Ked10b, Thm. 11.3.2].

Notes: the super-harmonicity is stated in [Ked10b] for a point of type (2) i.e. such that $\Delta(\xi)$ has at least two elements. If $\Delta(\xi)$ has a unique element, then the super-harmonicity is just the log-concavity which is the claim i). directional finiteness is not mentioned in [Ked10b], but from

¹³Here the equality is intended with multiplicities i.e. if a slope s appears n_i -times in $NP^{\text{conv}}(M_i, \xi)$, then it appears $n_1 + n_2$ -times in $NP^{\text{conv}}(M, \xi)$.

¹⁴Because of Lemma 3.4 one only controls the slopes along K -rational directions. This is the reason of taking $K = \widehat{K^{\text{alg}}}$.

¹⁵Some of these results were previously proved in [CD94, Th.2.5], and [Pon00].

the proof of [Ked10b, Thm. 11.3.2] it follows that the log-slopes $\partial_- H_{i,\delta}^{M,\text{sp}}(\xi_{t,\bar{p}})$ are those of a convenient differential polynomial with coefficients in the fraction field $\mathcal{F}(X)$ of $\mathcal{O}(X)$ so the directional finiteness (C5) holds applying Proposition 4.2, after possibly replacing M with its restriction to a sub-affinoid X' of X preserving the directions at $\xi_{t,\bar{p}}$ (cf. Def. 5.8) and such that the coefficients of the polynomial have all no poles in X' (cf. section 5.4.1). Finally notice that the harmonicity statement of [Ked10b, 11.3.2,(c)] implicitly assumes $K = \widehat{K^{\text{alg}}}$ because Lemma 3.4 is used to control the slopes along the branches $\Lambda(t)$ defined by algebraic $t \in K^{\text{alg}}$. \square

REMARK 5.7. We will reproduce and generalize a part of Theorem 5.6 (cf. section 7).

5.3 Restriction to a sub-affinoid

Let $X' \subseteq X$ be a sub-affinoid domain. The polygon $NP^{\text{sp}}(M, \xi)$ only depends on the restricted module $M_{\xi} = M \widehat{\otimes} \mathcal{H}(\xi)$, so it is invariant by restriction to X' . Conversely the slopes of $NP^{\text{conv}}(M, \xi)$ are upper bounded by $\rho_{\xi,X}$, and so the convergence polygon is not stable by restriction to X' .

DEFINITION 5.8. A sub-affinoid $X' \subseteq X$ preserves the directions at $\xi \in X$ if $\xi \in X'$, and if none of the directions in $\Delta(\xi)$ nor the extra direction toward $+\infty$ is suppressed when intersecting with X' .

REMARK 5.9. If $X' \subseteq X$ preserves the directions at $\xi \in X$, then $\xi \in X_{\text{int}}$ (resp. ξ is not in the Shilov boundary of X) if and only if $\xi \in X'_{\text{int}}$ (resp. ξ is not in the Shilov boundary of X').

We preserve the conventions of Remark 2.5.¹⁶

PROPOSITION 5.10. Let $X' \subset X$ be a sub-affinoid. Let $M' := M \otimes_{\mathcal{O}(X)} \mathcal{O}(X')$. Then

- i) For all $i = 1, \dots, r$ and all $\xi' \in X'$ one has $\mathcal{R}_i^{M'}(\xi') = \min(\mathcal{R}_i^M(\xi'), \rho_{\xi',X'})$, and

$$\Gamma(X', \mathcal{R}_i^{M'}) = \left(\Gamma(X, \mathcal{R}_i^M) \cap X' \right) \cup \Gamma_{X'}. \quad (5.3)$$

- ii) \mathcal{R}_i^M is directional finite at $\xi' \in X'$ (cf. (C5)) if and only if $\mathcal{R}_i^{M'}$ is directional finite at ξ' .
 iii) If $\Gamma_{X'} \subseteq \Gamma(X, \mathcal{R}_i^M)$, then for all $\xi'' \in X'$ one has $\mathcal{R}_i^{M'}(\xi'') = \mathcal{R}_i^M(\xi'')$ and $H_i^{M'}(\xi'') = H_i^M(\xi'')$. If moreover X' preserves the directions at $\xi' \in X'$, and if $K = \widehat{K^{\text{alg}}}$, then H_i^M is super-harmonic (resp. harmonic) at ξ' if and only if so does $H_i^{M'}$ at ξ' .
 iv) Assume that X' preserves the directions at $\xi' \in X'$, and that $\mathcal{R}_i^M(\xi') < \rho_{\xi',X'}$. Then for all $j = 1, \dots, i$ and all $\delta \in \Delta(\xi')$ one has $\partial_- \mathcal{R}_{j,\delta}^M(\xi') = \partial_- \mathcal{R}_{j,\delta}^{M'}(\xi')$, and hence $\partial_- H_{j,\delta}^M(\xi') = \partial_- H_{j,\delta}^{M'}(\xi')$. The same holds for the right slopes. Hence if $K = \widehat{K^{\text{alg}}}$, then H_j^M is super-harmonic (resp. harmonic) at ξ' if and only if so does $H_j^{M'}$ at ξ' .

Proof. Let $t \in X(\Omega)$ be a Dwrok generic points for ξ' . Clearly $\text{Sol}(M', t, \Omega) = \text{Sol}(M, t, \Omega)$, and if $R \leq \rho_{\xi',X'}$, then $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) = \text{Fil}^{\geq R} \text{Sol}(M', t, \Omega)$. Moreover $\text{Fil}^{\geq R} \text{Sol}(M, t, \Omega) \subseteq \text{Fil}^{\geq \rho_{\xi',X'}} \text{Sol}(M', t, \Omega)$ for all $\rho_{\xi',X'} \leq R \leq \rho_{\xi',X}$. This proves that the convergent slope sequence of M' at ξ' equals that of M truncated by $\rho_{t,X'}$. In other words $\mathcal{R}_i^{M'}(\xi') = \min(\mathcal{R}_i^M(\xi'), \rho_{\xi',X'})$ for all $\xi' \in X'$. From this expression together with (4.9) one sees that $\rho_{\mathcal{R}_i^{M'}}(\xi') = \min(\rho_{\mathcal{R}_i^M}(\xi'), \rho_{\xi',X'})$. This implies (5.3), and hence ii). If $\Gamma_{X'} \subseteq \Gamma(X, \mathcal{R}_i^M)$, then $\mathcal{R}_i^M(\xi') \leq \rho_{\mathcal{R}_i^M}(\xi') = \rho_{\Gamma(X, \mathcal{R}_i^M)}(\xi') \leq \rho_{\Gamma_{X'}}(\xi') = \rho_{\xi',X'}$ (cf. point v) of Prop. 2.2), and $\mathcal{R}_i^{M'}(\xi') = \mathcal{R}_i^M(\xi')$ for all $\xi' \in X'$. The same holds for all $j \leq i$ because $\mathcal{R}_j^M(\xi') \leq \mathcal{R}_i^M(\xi') \leq \rho_{\xi',X'}$. This proves that $H_i^{M'}(\xi') = H_i^M(\xi')$. This implies the assertion about the (super-)harmonicity since the equality of the functions implies the equality of the slopes. To prove iv) observe that $\mathcal{R}_j^{M'}(\xi') = \min(\mathcal{R}_j^M(\xi'), \rho_{\xi',X'}) = \mathcal{R}_j^M(\xi')$ since $\mathcal{R}_j^M(\xi') \leq \mathcal{R}_i^M(\xi') < \rho_{\xi',X'}$, and that this remains true by continuity on a open segment containing ξ' of each direction $\delta \in \Delta(\xi')$. \square

¹⁶Note that $\mathcal{R}_i^{M'}$ is not the restriction to \mathcal{R}_i^M to X' , so Remark 2.5 does not applies here.

5.4 Base change by a matrix in the fraction field $\mathcal{F}(X)$ of $\mathcal{O}(X)$

Let $\mathcal{F}(X)$ denotes the fraction field of $\mathcal{O}(X)$. If $H(T) \in GL_n(\mathcal{F}(X))$ is a matrix, its entries and those of its inverse have a finite number of poles in X , and these poles are algebraic over K . Then there exists a sub-affinoid $X' \subseteq X$ having conveniently small holes around the zeros and poles of $H(T)$ and its inverse, in order that $H \in GL_n(\mathcal{O}(X'))$. If $\xi \in X_{\text{int}}$, then ξ can not be a zero of $H(T)$, and X' can be chosen in order to preserve the directions at ξ (cf. Def. 5.8).

5.4.1 Reduction to a cyclic module. Let $r := \text{rk}(M)$ be the rank of M . By the cyclic vector theorem (cf. [Kat87]) one finds a cyclic basis of $M \otimes_{\mathcal{O}(X)} \mathcal{F}(X)$ in which M is represented by an operator $\mathcal{L} := \sum_{i=0}^r g_{r-i}(T)(d/dT)^i$, with $g_i \in \mathcal{F}(X)$ for all i , and $g_0 = 1$. Then \mathcal{L} represents simultaneously each $M_\xi = M \otimes_{\mathcal{O}(X)} \mathcal{H}(\xi)$ for all $\xi \in X^{\text{gen}}$. If $H(T) \in \mathcal{F}(X)$ is the base change matrix, one can chose $X' \subseteq X$ as indicated in section 5.4. We further restrict X' in order that none of the g_i has poles nor zeros on it. By Proposition 5.10 the restriction of M to X' does not affect the finiteness. If moreover $\Gamma_{X'} \subseteq \Gamma(\mathcal{R}_i^M)$, the super-harmonicity of H_i^M is also preserved.

6. Pull-back and push-forward by Frobenius

In this section K is assumed of mixed characteristic $(0, p)$ with $p > 0$. We assume moreover $\mu_p(\widehat{K^{\text{alg}}}) = \mu_p(K)$. If $\alpha, \alpha' \in \mu_p(K)$ are two distinct p -th root of 1, then $|\alpha - \alpha'| = \omega = |p|^{\frac{1}{p-1}}$. Let T, \tilde{T} be two variables. The ring morphism $\varphi^\# : K[T] \rightarrow K[\tilde{T}]$ sending $f(T)$ into $f(\tilde{T}^p)$, defines a analytic map $\varphi : \mathbb{A}_K^{1, \text{an}} \rightarrow \mathbb{A}_K^{1, \text{an}}$. If $t \in \mathbb{A}_K^{1, \text{an}}(\Omega)$ is a Dwork generic point for $\xi \in \mathbb{A}_K^{1, \text{an}}$, then t^p is a Dwork generic point for $\varphi(\xi)$. Indeed for all $f \in K[T]$ one has $\varphi(\xi)(f) = \xi(f(\tilde{T}^p)) = |f(t^p)|_\Omega$. We now describe the image of a point of type $\xi_{t, \rho}$. For all $h > 0$ and $\rho, \rho' \geq 0$ we set

$$\phi(h, \rho) := \max(\rho^p, |p|h^{p-1}\rho) = \begin{cases} \rho^p & \text{if } \rho \geq \omega \cdot h \\ |p|h^{p-1}\rho & \text{if } \rho \leq \omega \cdot h \end{cases}, \quad (6.1)$$

$$\psi(h, \rho') := \min\left((\rho')^{1/p}, \frac{\rho'}{|p|h^{p-1}}\right) = \begin{cases} (\rho')^{1/p} & \text{if } \rho' \geq \omega^p \cdot h^p \\ \frac{\rho'}{|p|h^{p-1}} & \text{if } \rho' \leq \omega^p \cdot h^p \end{cases}. \quad (6.2)$$

For h fixed ϕ and ψ are increasing functions such that $\phi(h, \psi(h, \rho')) = \rho'$ and $\psi(h, \phi(h, \rho)) = \rho$. In the sequel of this section by convention one sets $\rho' = \phi(h, \rho)$ and $\rho = \psi(h, \rho')$.

PROPOSITION 6.1. *Let $t \in \Omega$, $\rho > 0$. Then $\varphi(\xi_{t, \rho}) = \xi_{t^p, \phi(|t|, \rho)}$, hence $\varphi^{-1}(\xi_{t^p, \rho'}) = \{\xi_{\alpha t, \psi(|t|, \rho')}\}_{\alpha^p=1}$.*

Proof. By density and by multiplicativity it is enough to prove that for all $a \in K$ one has $\varphi(\xi_{t, \rho})(T - a) = \xi_{t^p, \phi(|t|, \rho)}(T - a)$. Write $\varphi(\xi_{t, \rho})(T - a) = \xi_{t, \rho}(\tilde{T}^p - a) = \xi_{t, \rho}((\tilde{T} - t + t)^p - a) = \xi_{t, \rho}(\sum_{k=0}^p \binom{p}{k} (\tilde{T} - t)^k t^{p-k} - a) = \max(|t^p - a|, |p||t|^{p-1}\rho, \rho^p)$, in fact the terms corresponding to $k = 1, \dots, p-1$ form either a non decreasing or a non increasing sequence. On the other hand $\xi_{t^p, \phi(|t|, \rho)}(T - a) = \xi_{t^p, \phi(|t|, \rho)}(T - t^p + t^p - a) = \max(\phi(|t|, \rho), |t^p - a|) = \varphi(\xi_{t, \rho})(T - a)$. \square

REMARK 6.2. *If t is a Dwork generic point for $\xi \in X^{1/p}$, then $|t| = \xi(\tilde{T})$ is independent on t .*

REMARK 6.3. *If $\rho = \psi(|t|, \rho') \geq \omega|t| = |\alpha - \alpha'||t|$, then $\xi_{\alpha t, \psi(|t|, \rho')} = \xi_{\alpha' t, \psi(|t|, \rho')}$ for all $\alpha, \alpha' \in \mu_p(K)$. Hence $\varphi^{-1}(\xi_{t^p, \rho'})$ has a single element. Conversely if $\rho < \omega|t|$, then $\varphi^{-1}(\xi_{t^p, \rho'})$ has p distinct elements.*

PROPOSITION 6.4. *Let $t \in \Omega$ and let $\rho, \rho' \geq 0$ be such that $\rho = \psi(|t|, \rho')$ and $\rho' = \phi(|t|, \rho)$. Let $D^-(t, \rho)$ and $D^-(t^p, \rho')$ be the open disks with algebras $\mathcal{A}_\Omega(t, \rho)$ and $\mathcal{A}_\Omega(t^p, \rho')$ respectively. Then:*

i) *One has the following equalities*

$$\varphi(D^-(t, \rho)) = D^-(t^p, \phi(|t|, \rho)), \quad \varphi^{-1}(D^-(t^p, \rho')) = \cup_{\alpha^p=1} D^-(\alpha t, \psi(|t|, \rho')) \quad (6.3)$$

- ii) For all $\alpha \in \mu_p(K)$ the corresponding morphism $\varphi_{\alpha,\rho}^\# : \mathcal{A}_\Omega(t^p, \rho') \rightarrow \mathcal{A}_\Omega(\alpha t, \psi(|t|, \rho'))$ is injective and isometric in the following sense. For all $f \in \mathcal{A}_\Omega(t^p, \rho')$ and all $\eta < \rho'$ one has

$$|f|_{t^p, \eta} = |\varphi_{\alpha,\rho}^\#(f)|_{\alpha t, \psi(|t|, \eta)}. \quad (6.4)$$

- iii) If $\rho' \leq \omega^p |t|^p$, then for all $\alpha \in \mu_p(K)$, $\varphi_{\alpha,\rho}^\#$ is an isomorphism of rings (satisfying (6.4)).
 iv) If $\omega^p |t|^p < \rho'$, then $\varphi_\rho^\# := \varphi_{\alpha,\rho}^\#$ is independent on α . Moreover $\mu_p(K)$ acts on $\mathcal{A}_\Omega(t, \rho)$ by $\alpha(f)(\tilde{T}) := f(\alpha \tilde{T})$, and

$$\varphi_\rho^\#(\mathcal{A}_\Omega(t^p, \rho')) = \mathcal{A}_\Omega(t, \rho)^{\mu_p(K)}. \quad (6.5)$$

- v) For all $\rho \geq 0$ we denote by $\psi_{\alpha,\rho}^\# : \mathcal{A}_\Omega(\alpha t, \rho) \rightarrow \mathcal{A}_\Omega(t^p, \rho')$ the Ω -linear map defined as $\psi_{\alpha,\rho}^\# := \begin{cases} (\varphi_{\alpha,\rho}^\#)^{-1} & \text{if } 0 \leq \rho \leq \omega |t| \\ \frac{1}{p} \sum_{\alpha^p=1} f(\alpha \tilde{T}) & \text{if } \omega |t| < \rho \end{cases}$. Then for all $\alpha \in \mu_p(K)$ the maps $\{\psi_{\alpha,\rho}^\#\}_{\rho \geq 0}$ satisfy $\psi_{\alpha,\rho}^\# \circ \varphi_{\alpha,\rho}^\# = \text{Id}_{\mathcal{A}_\Omega(t^p, \rho')}$ for all $\rho \geq 0$. Moreover if $\omega |t| \notin [\rho_1, \rho_2]$, and if $\rho'_i := \phi(|t|, \rho_i)$, $i = 1, 2$, then the following diagram is commutative where the horizontal maps are the restrictions:

$$\begin{array}{ccc} \mathcal{A}_\Omega(\alpha t, \rho_1) & \longleftarrow & \mathcal{A}_\Omega(\alpha t, \rho_2) \\ \psi_{\alpha,\rho_1}^\# \downarrow & \circlearrowleft & \downarrow \psi_{\alpha,\rho_2}^\# \\ \mathcal{A}_\Omega(t^p, \rho'_1) & \longleftarrow & \mathcal{A}_\Omega(t^p, \rho'_2). \end{array} \quad (6.6)$$

If $\omega |t| \in [\rho_1, \rho_2]$, then the diagram does not commute.

Proof. To prove i) one needs to evaluate $|a^p - t^p|$ for all $a \in D_{\Omega'}^-(t, \rho)$, and arbitrary $\Omega' \in E(\Omega)$. The skeleton of $f(\tilde{T}) = \tilde{T}^p - t^p = \prod_{\alpha^p=1} (\tilde{T} - \alpha t)$ is $\Gamma(f) = \text{Sat}(\{\alpha t\}_{\alpha^p=1})$. Hence for all $a \in \Omega$ one has $|a^p - t^p| = |\tilde{T}^p - t^p|_{a, |a - \alpha_a t|} |\tilde{T}^p - t^p|_{\alpha_a t, |a - \alpha_a t|} = \phi(|t|, |a - \alpha_a t|)$, where $\alpha_a \in \mu_p(K)$ satisfies $|a - \alpha_a t| = \min_{\alpha^p=1} |a - \alpha t|$. In fact by Prop. 6.1 one has $|\tilde{T}^p - t^p|_{\alpha t, \rho} = \phi(|t|, \rho)$. Since $\rho \mapsto \varphi(|t|, \rho)$ is strictly increasing, and since $|a - \alpha_a t| \leq |a - t|$, then $|a^p - t^p| \leq \phi(|t|, |a - t|)$. This proves $\varphi(D^-(t, \rho)) \subseteq D^-(t^p, \phi(|t|, \rho))$. Conversely applying $\psi(|t|, -)$ to $|a^p - t^p| = \varphi(|t|, |a - \alpha_a t|)$ with $b := a^p$ one finds $\psi(|t|, |b - t^p|) = |\alpha' b^{1/p} - \alpha' \alpha_{b^{1/p}} t|$ for all $\alpha' \in \mu_p(K)$. So i) holds. To prove ii), by density one can assume that f is a polynomial in $\Omega[T] \subset \mathcal{A}_\Omega(t^p, \rho')$, and by multiplicativity up to enlarge Ω one can assume $f = (T - a)$ of degree 1. In this case the assertion is easy. The injectivity follows from 6.4. To prove iii) we can assume $t \neq 0$ because $\rho \leq \omega^p |t|^p$. We need the following

LEMMA 6.5. Let $t \neq 0$ and $\alpha \in \mu_p(K)$. For all $k = 1, \dots, p-1$ the power series $T^{k/p} - \alpha t = (T - t^p + t^p)^{k/p} - \alpha t = (t^k - \alpha t) + t^k \cdot \sum_{s \geq 1} \binom{k/p}{s} \left(\frac{T - t^p}{t^p}\right)^s$ has radius of convergence $\omega^p |t|^p$ around t^p .

Proof. Since $|\binom{k/p}{s}| = |k/p|^s / |s|!$, then $\liminf_s |\binom{k/p}{s} t^{-ps}|_\Omega^{-1/s} = |p| |t|^p \liminf_s |s|^{1/s} = \omega^p |t|^p$. \square

We now define $(\varphi_{\alpha,\rho}^\#)^{-1}$. By Lemma 6.5, $g := T^{1/p} - \alpha t \in \mathcal{A}_\Omega(t^p, \rho')$ because $\rho' \leq \omega^p |t|^p$. Hence $(\varphi_{\alpha,\rho}^\#)^{-1}(\sum_{i=0}^n a_i (\tilde{T} - \alpha t)^i) = \sum_{i=0}^n a_i g^i$. This defines a map $\Omega[\tilde{T} - t] \rightarrow \mathcal{A}_\Omega(t^p, \rho')$ satisfying (6.4), that extends by continuity to $\mathcal{A}_\Omega(t, \rho)$ and coincides with $(\varphi_{\alpha,\rho}^\#)^{-1}$. This proves iii). Assume $\omega |t| < \rho$. Since $\Omega[\tilde{T}]^{\mu_p} = \Omega[\tilde{T}^p] = \varphi_\rho^\#(\Omega[T])$, then by density one has iv) together with $\psi_{\alpha,\rho}^\# \circ \varphi_{\alpha,\rho}^\# = \text{Id}_{\mathcal{A}_\Omega(t^p, \rho')}$. The commutativity of the diagram is evident. If $\omega |t| \in [\rho_1, \rho_2]$ it does not commute since for example $\psi_{\alpha,\rho_2}^\#(\tilde{T}) = 0 \neq \psi_{\alpha,\rho_1}^\#(\tilde{T})$, because $\psi_{\alpha,\rho_1}^\# = (\varphi_{\alpha,\rho_1}^\#)^{-1}$. \square

6.0.2 *Exact radius of the pull-back of a power series.* Let as above $\Omega \in E(K)$, $t \in \Omega$, and $\rho' \geq 0$. Let $f(T) = \sum_{s \geq 0} a_s (T - t^p)^s \in \mathcal{A}_\Omega(t^p, \rho')$ be a power series converging around t^p with exact radius $R_{f(T)} := \liminf_s |a_s|^{-1/s} \geq \rho'$. By Proposition 6.4 $f(\tilde{T}^p) = \sum_{s \geq 0} a_s (\tilde{T}^p - t^p)^s = \sum_s b_s (\tilde{T} - \alpha t)^s \in \mathcal{A}_\Omega(\alpha t, \psi(|t|, \rho'))$ has a radius $R_{f(\tilde{T}^p)} := \liminf_s |b_s|^{-1/s}$ satisfying

$$R_{f(\tilde{T}^p)} \geq \psi(|t|, R_{f(T)}), \quad R_{f(T)} \leq \phi(|t|, R_{f(\tilde{T}^p)}). \quad (6.7)$$

REMARK 6.6. Corollary 6.7 below proves that (6.7) are equalities in the most part of cases. The inequality is due to the fact that $R_{f(\tilde{T}^p)}$ is the radius of the composite of f and \tilde{T}^p as power series, while $\psi(|t|, R_{f(T)})$ is the radius of their composite as functions. For instance, by Lemma 6.5, $T^{k/p} - t$ converges with exact radius $\omega^p |t|^p$, but its pull-back $\tilde{T}^k - t$ converges with infinite radius.

COROLLARY 6.7. With the above notations one has

- i) If $R_{f(T)} \neq \omega^p |t|^p$, then $R_{f(\tilde{T}^p)} \neq \omega |t|$ and $R_{f(\tilde{T}^p)} = \psi(|t|, R_{f(T)})$. In particular this equality holds if $R_{f(T)} = 0$ (then $R_{f(\tilde{T}^p)} = 0$) or if $R_{f(T)} = +\infty$ (then $R_{f(\tilde{T}^p)} = +\infty$).
- ii) If $R_{f(\tilde{T}^p)} \leq \omega |t|$, then $R_{f(T)} = \phi(|t|, R_{f(\tilde{T}^p)}) = |p| |t|^{p-1} R_{f(\tilde{T}^p)}$.
- iii) If $R_{f(\tilde{T}^p)} > \omega |t|$, then $R_{f(T)} \geq \omega^p |t|^p$.

Proof. One has $\phi(|t|, \omega |t|) = \omega^p |t|^p$ and $\psi(|t|, \omega^p |t|^p) = \omega |t|$. Assume that $R_{f(T)} \neq \omega^p |t|^p$ and, by contrapositive, that (6.7) is strict. Let $R_{f(\tilde{T}^p)} > \rho > \psi(|t|, R_{f(T)})$ be such that $\omega |t| \notin [\psi(|t|, R_{f(T)}), \rho]$. By diagram (6.6) $f(\tilde{T}^p) \in \mathcal{A}_\Omega(\alpha t, \rho)$ and $f(T) = \psi_{\alpha, \rho}^\#(f(\tilde{T}^p)) \in \mathcal{A}_\Omega(t^p, \varphi(|t|, \rho))$. Now $\varphi(|t|, \rho) > \varphi(|t|, \psi(|t|, R_{f(T)})) = R_{f(T)}$ which is a contradiction. ii) and iii) follows from Prop. 6.4, iii). \square

6.0.3 Degree of the residual fields. The morphism φ induces a K -linear isometric inclusion

$$\varphi^\# : \mathcal{H}(\varphi(\xi)) \rightarrow \mathcal{H}(\xi). \quad (6.8)$$

PROPOSITION 6.8. Let $\xi = \xi_{c, \rho}$, $c \in \Omega$, $\rho \geq r(\xi_c)$. Then :

- i) If $\omega |c| < \rho$, then $\mathcal{H}(\xi)/\mathcal{H}(\varphi(\xi))$ is an extension of degree p .
- ii) Conversely if $c \in K$, $c \neq 0$, and if $\rho < \omega |c|$, then $\mathcal{H}(\xi) = \mathcal{H}(\varphi(\xi))$.

Proof. $K(\tilde{T})$ (resp. $K(T)$) is dense on $\mathcal{H}(\xi)$ (resp. $\mathcal{H}(\varphi(\xi))$). The map $\varphi^\# : K(T) \rightarrow K(\tilde{T})$, $T \mapsto \tilde{T}^p$ is an extension of degree p . By density the degree of $\mathcal{H}(\xi)/\mathcal{H}(\varphi(\xi))$ is equal to 1 or p , and it is 1 if and only if $\tilde{T} = T^{1/p} \in \mathcal{H}(\varphi(\xi))$. One has $r(\xi_{c, \rho}) = \max(r(\xi_c), \rho) = \rho$, and $r(\varphi(\xi)) = r(\xi_{c^p, \phi(|c|, \rho)}) = \min(r(\xi_{c^p}), \phi(|c|, \rho)) = \min(\phi(|c|, r(\xi_c)), \phi(|c|, \rho)) = \phi(|c|, \rho)$. Let $\Omega \in E(K)$, $t = t_{c, \rho} \in \Omega$ (resp. $t^p \in \Omega$) be a Dwork generic point for $\xi_{c, \rho}$ (resp. $\xi_{c^p, \phi(|c|, \rho)}$). As in section 4.3.1 one has a diagram

$$\begin{array}{ccc} \mathcal{H}(\xi) & \longrightarrow & \mathcal{A}_\Omega(t_{c, \rho}, \rho) \\ \varphi^\# \uparrow & \circlearrowleft & \uparrow \varphi_{1, \rho}^\# \\ \mathcal{H}(\varphi(\xi)) & \longrightarrow & \mathcal{A}_\Omega(t_{c^p, \rho}^p, \phi(|c|, \rho)). \end{array} \quad (6.9)$$

If $\rho > \omega |c|$, then $|t_{c, \rho}| = |\tilde{T} - c + c|_{c, \rho} = \max(|c|, \rho) \geq \rho > \max(\omega |c|, \omega \rho) = \omega |t_{c, \rho}|$. And hence $\phi(|c|, \rho) = \phi(|t_{c, \rho}|, \rho) = \rho^p$. By contrapositive if $T^{1/p} \in \mathcal{H}(\varphi(\xi))$, then $T^{1/p} - t_{c, \rho} \in \mathcal{A}_\Omega(t_{c^p, \rho}^p, \rho^p)$ which is absurd by Lemma 6.5. So $[\mathcal{H}(\xi) : \mathcal{H}(\varphi(\xi))] = p$. To prove ii) write $T^{1/p} - c = \lim_s c \sum_{k=1}^s \binom{1/p}{k} \left(\frac{T - c^p}{c^p}\right)^k$. This limit converges in $\mathcal{H}(\varphi(\xi))$ with respect to $\varphi(\xi) = \xi_{c^p, |p||c|^{p-1}\rho}$. Indeed $|\binom{1/p}{k} (\frac{T - c^p}{c^p})^k|_{c^p, |p||c|^{p-1}\rho} = \frac{|1/p|^k}{|k!|} \frac{|p|^k \rho^k}{|c|^k} = \frac{\rho^k}{|c|^k |k!|} \rightarrow 0$ since $\rho < \omega |c|$. Hence $T^{1/p} \in \mathcal{H}(\varphi(\xi))$. \square

6.1 Behavior of the radii of convergence under pull-back by Frobenius

Let $X \subset \mathbb{A}_K^{1, \text{an}}$ and $X^{1/p} := \varphi^{-1}(X)$. The image of the injective map $\varphi^\# : \mathcal{O}(X) \rightarrow \mathcal{O}(X^{1/p})$ is not stable under $d/d\tilde{T}$. One has $(\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}}) \circ \varphi^\# = \varphi^\# \circ d/dT$, so one is induced to assume that $0 \notin X$ (hence $0 \notin X^{1/p}$) in sections 6.1 and 6.3. Then $p\tilde{T}^{p-1} \in \mathcal{O}(X^{1/p})^\times$ and $\varphi^\# : (\mathcal{O}(X), d/dT) \rightarrow (\mathcal{O}(X^{1/p}), \frac{d/d\tilde{T}}{p\tilde{T}^{p-1}})$ commutes with the derivations. The pull-back by Frobenius φ^* is the composite

$$\frac{d}{dT} - \text{Mod}(\mathcal{O}(X)) \longrightarrow \left(\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}}\right) - \text{Mod}(\mathcal{O}(X^{1/p})) \xrightarrow{\sim} \frac{d}{d\tilde{T}} - \text{Mod}(\mathcal{O}(X^{1/p})) \quad (6.10)$$

where the first functor is the usual scalar extension functor associating to (M, ∇) the $\mathcal{O}(X^{1/p})$ -differential module $\tilde{M} = M \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{1/p})$ together with $\tilde{\nabla} := \nabla \otimes 1 + 1 \otimes (\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}})$, which is a connection with respect to the derivation $\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}}$ of $\mathcal{O}(X^{1/p})$. The second functor is an equivalence of categories that only changes the derivation: it sends $(\tilde{M}, \tilde{\nabla})$ into $(\tilde{M}, p\tilde{T}^{p-1}\tilde{\nabla})$ and it is the identity on the morphisms.¹⁷ Concretely if $\frac{d}{dT}(Y) = G(T)Y$ is the equation in a basis e of M , then $\frac{d}{d\tilde{T}}(Y) = p\tilde{T}^{p-1}G(\tilde{T})Y$ is that of $\varphi^*(M) := M \otimes_{\mathcal{O}(X)} \mathcal{O}(X^{1/p})$ in the basis $e \otimes 1$. If $Y(T)$ is a Taylor solution of $Y' = GY$, then $Y(\tilde{T})$ is the Taylor solution of $\frac{d}{d\tilde{T}}(Y) = p\tilde{T}^{p-1}G(\tilde{T})Y$.

PROPOSITION 6.9. *Let M be a differential module over $\mathcal{O}(X)$ of rank r . Let $t \in X^{1/p}(\Omega)$ be a Dwork generic point for $\xi \in X^{1/p}$. Then by (6.7) one has $\mathcal{R}_1^{\varphi^*M}(\xi) \geq \psi(|t|, \mathcal{R}_1^M(\xi))$. Moreover :*

- i) *If for all $i = 1, \dots, i_0$ one has $\mathcal{R}_i^M(\varphi(\xi)) \neq \omega^p|t|^p$, then $\mathcal{R}_{i_0}^{\varphi^*M}(\xi) = \psi(|t|, \mathcal{R}_{i_0}^M(\varphi(\xi)))$.*
- ii) *If $\mathcal{R}_i^{\varphi^*M}(\xi) \leq \omega|t|$, then $\mathcal{R}_i^M(\varphi(\xi)) = \phi(|t|, \mathcal{R}_i^{\varphi^*M}(\xi)) = |p||t|^{p-1}\mathcal{R}_i^{\varphi^*M}(\xi)$.*

Proof. The map $\text{Id} \otimes \varphi^\# : M \otimes \mathcal{A}_\Omega(t^p, \rho') \rightarrow M \otimes \mathcal{A}_\Omega(t, \psi(|t|, \rho'))$ induces an Ω -linear isomorphism $\text{Sol}(M, t^p, \Omega) \xrightarrow{\sim} \text{Sol}(\varphi^*M, t, \Omega)$. By Proposition 6.4 one has $\rho_{\xi, X^{1/p}} = \phi(|t|, \rho_{\varphi(\xi), X})$. So, for all $\rho' \leq \rho_{\varphi(\xi), X}$, identifying $M \otimes \mathcal{A}_\Omega(t^p, \rho') \xrightarrow{\sim} \mathcal{A}_\Omega(t^p, \rho')^r$ a solution $\tilde{y}(T) \in \mathcal{A}_\Omega(t^p, \rho')^r$ is sent into $\tilde{y}(\tilde{T}) \in \mathcal{A}_\Omega(t, \psi(|t|, \rho'))^r$. Corollary 6.7 gives $\text{Id} \otimes \varphi^\#(\text{Fil}^{\geq \rho'} \text{Sol}(M, t^p, \Omega)) \subseteq \text{Fil}^{\geq \psi(|t|, \rho')} \text{Sol}(\varphi^*M, t, \Omega)$. The only “pathology” that can happens is that a solution $\tilde{y}(T)$ converging with exact radius $\omega^p|t|^p$ is sent into a solution $\tilde{y}(\tilde{T})$ with radius strictly larger than $\omega|t| = \psi(|t|, \omega^p|t|^p)$. This increase the dimension of $\text{Fil}^{\geq \psi(|t|, \rho')} \text{Sol}(\varphi^*M, t, \Omega)$ with respect to that of $\text{Fil}^{\geq \rho'} \text{Sol}(M, t^p, \Omega)$. For $\rho' \leq \omega^p|t|^p$ the dimensions are equal, so i) and ii) hold. For $\rho' > \omega^p|t|^p$. The assumption i) implies that there is no “pathology”, so all the radii are transformed by the rule $\psi(|t|, -)$, and the dimensions are equal. \square

COROLLARY 6.10. *Assume $\xi \in X_{\text{int}}$ is of the form $\xi_{t, \rho}$ with $t \in K$, and $\rho > 0$. Then Proposition 6.9 holds replacing $\mathcal{R}_i^M(\xi)$ and $\mathcal{R}_i^{\varphi^*M}(\xi)$ by $\mathcal{R}_i^{M, \text{sp}}(\xi)$ and $\mathcal{R}_i^{\varphi^*M, \text{sp}}(\xi)$ respectively.*

Proof. The claim follows by truncation from point i) of Theorem 4.7 (cf. section 4.3.1). \square

6.2 Antecedent by Frobenius.

PROPOSITION 6.11. *Let $t \in \Omega$ be a Dwork generic point for $\xi \in (\mathbb{A}_K^{1, \text{an}})^{\text{gen}}$. Let \tilde{M} be a $\mathcal{H}(\xi)$ -differential module satisfying $\mathcal{R}^{\tilde{M}, \text{sp}}(\xi) > \omega|t| = \omega\xi(\tilde{T})$. Then there exists a unique $\mathcal{H}(\varphi(\xi))$ -differential module M satisfying both conditions $\varphi^*(M) \cong \tilde{M}$ and $\mathcal{R}^{M, \text{sp}}(\varphi(\xi)) = \mathcal{R}^{\tilde{M}, \text{sp}}(\xi)^p$.*

Proof. The proof is the same as [Ked10b, Thm.10.4.2]. \square

6.3 Definition of the push-forward by Frobenius

If $\varphi^\# : \mathcal{H}(\varphi(\xi)) \xrightarrow{\sim} \mathcal{H}(\xi)$ is an isomorphism, then the Frobenius pull-back φ^* is an equivalence, and the push-forward φ_* is by definition its quasi inverse. We then consider point i) of Prop. 6.8:

HYPOTHESIS 6.12. *In this section we assume $\xi = \xi_{c, \rho}$, with $c \in \Omega$, $\rho \geq r(\xi_c)$ and $\rho > \omega|c|$.*

REMARK 6.13. *Under 6.12 one has $\varphi(\xi) = \varphi(\xi_{c, \rho}) = \xi_{c^p, \rho^p}$. If one needs to perform n -times the Frobenius push-forward, then one has to assume $\rho > (\omega|c|)^{1/p^n}$ to have $[\mathcal{H}(\xi) : \mathcal{H}(\varphi^n(\xi))] = p^n$.*

¹⁷Namely if $\tilde{\nabla} : \tilde{M} \rightarrow \tilde{M}$ is a connection with respect to $\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}}$, then $p\tilde{T}^{p-1}\tilde{\nabla} : \tilde{M} \rightarrow \tilde{M}$ is a connection with respect to $d/d\tilde{T}$, and one sees that an $\mathcal{O}(X^{1/p})$ -linear morphism commutes with $\tilde{\nabla}$ s if and only if it commutes with $p\tilde{T}^{p-1}\tilde{\nabla}$ s.

The *Frobenius push-forward functor* φ_* is the composite functor

$$\frac{d}{d\tilde{T}} - \text{Mod}(\mathcal{H}(\xi)) \xrightarrow{\sim} \left(\frac{d/d\tilde{T}}{p\tilde{T}^{p-1}}\right) - \text{Mod}(\mathcal{H}(\xi)) \longrightarrow \frac{d}{dT} - \text{Mod}(\mathcal{H}(\varphi(\xi))) \quad (6.11)$$

where the first equivalence is the inverse of the change of derivation functor (6.10) and it associates to the $(\mathcal{H}(\xi), d/d\tilde{T})$ -differential module $(\tilde{M}, \tilde{\nabla})$ the $(\mathcal{H}(\xi), \frac{d/d\tilde{T}}{p\tilde{T}^{p-1}})$ -differential module $(\tilde{M}, \frac{\tilde{\nabla}}{p\tilde{T}^{p-1}})$. The second is the *scalar restriction* functor associating to $(\tilde{M}, \frac{\tilde{\nabla}}{p\tilde{T}^{p-1}})$ the $(\mathcal{H}(\varphi(\xi)), d/dT)$ -differential module $(\tilde{M}, \frac{\tilde{\nabla}}{p\tilde{T}^{p-1}})$ itself viewed as a $\mathcal{H}(\varphi(\xi))$ -module via $\varphi^\#$. We denote by $(\varphi_*(\tilde{M}), \varphi_*(\tilde{\nabla}))$ the differential module over $\mathcal{H}(\varphi(\xi))$ so obtained.

6.3.1 Matrix of $\varphi_*(\tilde{\nabla})$. One has a direct sum decomposition $\mathcal{H}(\xi) = \bigoplus_{k=0}^{p-1} \varphi^\#(\mathcal{H}(\varphi(\xi))) \cdot \tilde{T}^k$, so that each $g(\tilde{T}) \in \mathcal{H}(\xi)$ can be uniquely written as $g(\tilde{T}) = \sum_{k=0}^{p-1} g_k(\tilde{T}) \tilde{T}^k = \sum_{k=0}^{p-1} g_k(T) \tilde{T}^k$. $\frac{d}{d\tilde{T}} = \frac{1}{p\tilde{T}^{p-1}} \frac{d}{dT}$ stabilizes globally each factor and $\frac{1}{p\tilde{T}^{p-1}} \frac{d}{dT}(g_k(T) \tilde{T}^k) = g'_k(T) \tilde{T}^k$. For all $g(\tilde{T}) \in \mathcal{H}(\xi)$ we define $\varphi_*(g)(T) \in M_{p \times p}(\mathcal{H}(\varphi(\xi)))$ to be the matrix of the multiplication in $\mathcal{H}(\xi)$ by $g(\tilde{T})/(p\tilde{T}^{p-1})$, with respect to the basis $1, \tilde{T}, \dots, \tilde{T}^{p-1}$ over $\mathcal{H}(\varphi(\xi))$. One has

$$\varphi_*(g)(T) = (pT)^{-1} \cdot \begin{pmatrix} g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \dots & \dots & \dots & Tg_0(T) \\ g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \dots & \dots & Tg_1(T) \\ g_1(T) & g_0(T) & g_{p-1}(T) & Tg_{p-2}(T) & Tg_{p-3}(T) & \dots & Tg_2(T) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ g_{p-2}(T) & g_{p-3}(T) & \dots & \dots & g_1(T) & g_0(T) & g_{p-1}(T) \end{pmatrix} \quad (6.12)$$

Notice that the terms over the diagonal are multiplied by T . Let $(\tilde{M}, \tilde{\nabla})$ be a differential module over $\mathcal{H}(\xi)$. Fix a $\mathcal{H}(\xi)$ -linear isomorphism $\mathcal{H}(\xi)^r \xrightarrow{\sim} \tilde{M}$ (i.e. a basis of \tilde{M}), and let $\frac{d}{d\tilde{T}} - G(\tilde{T})$ be the map $\tilde{\nabla}$ in this basis. Writing $\mathcal{H}(\xi)^r = (\bigoplus_{k=0}^{p-1} \varphi^\#(\mathcal{H}(\varphi(\xi))) \cdot \tilde{T}^k)^r$ one sees that if $G(\tilde{T}) = (g_{i,j}(\tilde{T}))_{i,j=1,\dots,r} \in M_{r \times r}(\mathcal{H}(\xi))$, then the matrix of $\varphi_*(\tilde{\nabla})$ is given by the block matrix

$$\varphi_*(G)(T) := (\varphi_*(g_{i,j})(T))_{i,j=1,\dots,r} \in M_{pr \times pr}(\mathcal{H}(\varphi(\xi))). \quad (6.13)$$

6.4 The rank one modules $\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$.

We preserve the assumption 6.12. For all $k = 0, \dots, p-1$ the function $y_k := T^{k/p}$ verifies $\frac{d}{dT}(y_k) = \frac{k}{pT} y_k$. Let $\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$ be the corresponding differential module over $(\mathcal{H}(\varphi(\xi)), d/dT)$. Since $[\mathcal{H}(\xi) : \mathcal{H}(\varphi(\xi))] = p$ one has a decomposition of $\mathcal{H}(\varphi(\xi))$ -differential modules $\varphi_*(\mathcal{H}(\xi)) = \bigoplus_{k=0}^{p-1} \mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$. In the sequel we often identify \tilde{T} with $T^{1/p}$ and $\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$ with $\varphi^\#(\mathcal{H}(\varphi(\xi))) \cdot \tilde{T}^k \subseteq \mathcal{H}(\xi)$. For all $\mathcal{H}(\varphi(\xi))$ -differential module M we set $M[k] := M \otimes_{\mathcal{H}(\varphi(\xi))} \mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$.

LEMMA 6.14 [Ked10b, Lemma 10.3.6]. *One has the following properties:*

- i) For all $\mathcal{H}(\xi)$ -differential module \tilde{M} and all $k = 0, \dots, p-1$ one has a canonical isomorphism of $\mathcal{H}(\varphi(\xi))$ -differential modules $\mu_k : \varphi_*(\tilde{M})[k] \xrightarrow{\sim} \varphi_*(\tilde{M})$ defined by $m \otimes h(T) T^{k/p} \mapsto h(\tilde{T}^p) \tilde{T}^k m$.
- ii) An $\mathcal{H}(\varphi(\xi))$ -differential sub-module $N \subseteq \varphi_*(\tilde{M})$ is the push-forward of a submodule of \tilde{M} if and only if N viewed as a subgroup of \tilde{M} is stable under the multiplication by scalars of $\mathcal{H}(\xi)$. This happens if and only if for all $k = 0, \dots, p-1$ one has $\mu_k(N[k]) \subseteq N$.
- iii) For all differential module M over $\mathcal{H}(\varphi(\xi))$ one has $\varphi_* \varphi^* M \xrightarrow{\sim} \bigoplus_{k=0}^{p-1} M[k]$.
- iv) For all differential module \tilde{M} over $\mathcal{H}(\xi)$ one has $\varphi^* \varphi_* \tilde{M} \xrightarrow{\sim} \tilde{M}^{\oplus p}$. □

PROPOSITION 6.15. *Let t be a Dwork generic point for ξ . For all $k = 1, \dots, p-1$ one has $\mathcal{R}^{\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}}(\varphi(\xi)) = \omega^p |t|^p$.*

Proof. The solution of $y'_k(T) = \frac{k}{pT} y_k(T)$ around t^p is $T^{k/p}$. Then apply Lemma 6.5. □

6.5 Behavior of the radius by Frobenius push-forward

We preserve the assumption 6.12. Let t be a Dwork generic point for $\xi = \xi_{c,\rho}$.

PROPOSITION 6.16. *Let \tilde{M} be a differential module over $\mathcal{H}(\xi)$. Assume that $\mathcal{R}_{i_1}^{\tilde{M},\text{sp}}(\xi) \leq \omega|t| < \mathcal{R}_{i_1+1}^{\tilde{M},\text{sp}}(\xi)$. Then, up to a permutation, the list of the spectral radii of $\varphi_*\tilde{M}$ is given by*

$$\bigcup_{i \leq i_1} \underbrace{\left\{ |p||t|^{p-1} \mathcal{R}_i^{\tilde{M},\text{sp}}(\xi), \dots, |p||t|^{p-1} \mathcal{R}_i^{\tilde{M},\text{sp}}(\xi) \right\}}_{p\text{-times}} \bigcup_{i > i_1} \left\{ \mathcal{R}_i^{\tilde{M},\text{sp}}(\xi)^p, \underbrace{\omega^p|t|^p, \dots, \omega^p|t|^p}_{p-1\text{-times}} \right\}. \quad (6.14)$$

Proof. The proof comes from [Ked10b, Thm. 10.5.1] with slide modifications. We reproduce it for the convenience of the reader. We can assume that \tilde{M} has no non trivial sub-objects, so $\mathcal{R}_1^{\tilde{M},\text{sp}}(\xi) = \dots = \mathcal{R}_r^{\tilde{M},\text{sp}}(\xi) = R$. Assume $R > \omega|t|$. Let M be such that $\tilde{M} = \varphi^*(M)$ (cf. Prop. 6.11). Then M is simple, since a sub-object $N \subset M$ produces a sub-object φ^*N of \tilde{M} . Hence $\mathcal{R}_1^{M,\text{sp}}(\varphi(\xi)) = \dots = \mathcal{R}_r^{M,\text{sp}}(\varphi(\xi)) = R^p$. By Lemma 6.14 one has $\varphi_*\tilde{M} = \varphi_*\varphi^*M = \bigoplus_{k=0}^{p-1} M[k]$. Each $M[k] := M \otimes_{\mathcal{H}(\varphi(\xi))} \mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$ is irreducible since $\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}$ has rank one. Since $\mathcal{R}_1^{\mathcal{H}(\varphi(\xi)) \cdot T^{k/p}}(\varphi(\xi)) = \begin{cases} \omega^p|t|^p & \text{if } k \neq 0 \\ r(\varphi(\xi)) & \text{if } k = 0 \end{cases}$, and since $\omega^p|t|^p < R^p$, then $\mathcal{R}^{M[k],\text{sp}}(\varphi(\xi)) = \begin{cases} \omega^p|t|^p & \text{if } k \neq 0 \\ R^p & \text{if } k = 0 \end{cases}$, because the radius of a tensor product is the minimum of the radii if they are different (cf. [Ked10b, Lemma 9.4.6]). This proves the result. Assume now that $R \leq \omega|t|$. Let $R' := |p||t|^{p-1}R = \phi(|t|, R)$. One has a commutative diagram compatible with the derivations like (6.9) with $\mathcal{A}_\Omega(t_{c,\rho}, \rho)$ (resp. $\mathcal{A}_\Omega(t_{c,\rho}^p, \phi(|c|, \rho))$) replaced by $\mathcal{A}_\Omega(t, R)$ (resp. $\mathcal{A}_\Omega(t^p, R')$). The restriction $\mathcal{H}(\varphi(\xi)) \rightarrow \mathcal{A}_\Omega(t^p, R')$ then factorizes through $\mathcal{H}(\varphi(\xi)) \rightarrow \mathcal{H}(\xi) \rightarrow \mathcal{A}_\Omega(t, R) \xrightarrow{(\varphi_{1,R}^\#)^{-1}} \mathcal{A}_\Omega(t^p, R')$ (cf. Prop. 6.4, iii)). Hence $\varphi_*(\tilde{M}) \otimes_{\mathcal{H}(\varphi(\xi))} \mathcal{A}_\Omega(t^p, R') = \varphi^*\varphi_*(\tilde{M}) \otimes_{\mathcal{H}(\xi)} \mathcal{A}_\Omega(t^p, R') = \tilde{M}^p \otimes_{\mathcal{H}(\xi)} \mathcal{A}_\Omega(t^p, R')$ (cf. Lemma 6.14). By assumption \tilde{M} is trivialized by $\mathcal{A}_\Omega(t, R)$ and hence by $\mathcal{A}_\Omega(t^p, R')$, so $\mathcal{R}^{\varphi_*\tilde{M},\text{sp}}(\varphi(\xi)) \geq R'$. If now $0 \neq N \subseteq \varphi_*\tilde{M}$, then $\mathcal{R}^{N,\text{sp}}(\varphi(\xi)) \geq \mathcal{R}^{\varphi^*(N),\text{sp}}(\varphi(\xi)) \geq R'$. We claim that this is an equality for all N . Since $\varphi^*N \subseteq \varphi^*\varphi_*(\tilde{M}) = \tilde{M}^p$, each simple sub-quotient of φ^*N appears among those of \tilde{M} , and its radius is R . So $\mathcal{R}^{\varphi^*N,\text{sp}}(\xi) = R \leq \omega|t|$ and by Proposition 6.9 one has $\mathcal{R}^{N,\text{sp}}(\varphi(\xi)) = R'$. \square

6.6 Stability of the directional finiteness and harmonicity by Frobenius push-forward

An affinoid X is called a *pseudo-annulus* if Γ_X has, at most, a unique bifurcation point ξ , and if Γ_X has no punctured smooth points other than possibly ξ . A pseudo-annulus is obtained from an annulus $X' := \{|T - t| \in [R_1, R_0]\}$, $t \in K^{\text{alg}}$, by removing a finite number of disks $D^-(c_i, R_i)$, $c_i \in K^{\text{alg}}$, in order that the bifurcation point is $\xi = \xi_{t,\bar{\rho}}$ with $\bar{\rho} := |c_i - t| = |c_j - t|$ for all $i, j = 2, \dots, \mu$. We now assume $t = 0$ and $\xi = \xi_{0,\bar{\rho}}$. Let $X^p := \varphi(X)$. One defines as above the push-forward functor

$$\varphi_* : \frac{d}{d\tilde{T}} - \text{Mod}(\mathcal{O}(X)) \xrightarrow{\sim} \frac{d/d\tilde{T}}{p\tilde{T}^{p-1}} - \text{Mod}(\mathcal{O}(X)) \longrightarrow \frac{d}{dT} - \text{Mod}(\mathcal{O}(X^p)). \quad (6.15)$$

If $\xi' \in X$ verifies the assumption 6.12, then (6.15) is compatible with (6.11) at ξ' . By Proposition 6.1 the map $\varphi : X \rightarrow X^p$ identifies the directions through ξ and $\varphi(\xi)$:

$$\varphi : \Delta(\xi) \xrightarrow{\sim} \Delta(\varphi(\xi)), \quad \xi_{c,\rho} \mapsto \xi_{c^p,\rho^p}, \quad (6.16)$$

For ρ close to $\bar{\rho}$, $\xi_{c,\rho}$ verifies the assumption 6.12, and $[\mathcal{H}(\xi_{c,\rho}) : \mathcal{H}(\xi_{c^p,\rho^p})] = p$ by Prop. 6.8.

PROPOSITION 6.17. *Let X be a pseudo-annulus and $\xi := \xi_{0,\bar{\rho}}$ as above. Let \tilde{M} be a differential module over $\mathcal{O}(X)$. Let i_0 (resp. i_1) be the largest value of i such that $\mathcal{R}_i^{\tilde{M}}(\xi) < r(\xi) = \bar{\rho}$ (resp. $\mathcal{R}_i^{\tilde{M}}(\xi) \leq \omega r(\xi) = \omega\bar{\rho}$). For $i \leq i_0$ we define¹⁸ $\phi(i) := \begin{cases} pi & \text{if } 1 \leq i \leq i_1 \\ (p-1)r+i & \text{if } i_1 < i \leq i_0 \end{cases}$, $d_i := \begin{cases} i & \text{if } i \leq i_1 \\ r & \text{if } i_1 < i \leq i_0 \end{cases}$, and*

¹⁸It is understood that if $i_1 = i_0$, then $\phi(i) = pi$ and $d_i = i$ for all $i \in \{1, \dots, i_0\}$.

$f_i(\tilde{T}) := (p\tilde{T}^{(p-1)})^{d_i} \in \mathcal{O}(X)$. Let $i \in \{1, \dots, i_0\}$, then for all $c \in X(\Omega)$, $|c| \leq \bar{\rho}$, for all ρ close enough to $\bar{\rho}$, one has

$$H_i^{\tilde{M}}(\xi_{c,\rho}) = H_{\phi(i)}^{\varphi_* \tilde{M}}(\xi_{c^p,\rho^p})^{1/p} / |f_i(\xi_{c,\rho})|. \quad (6.17)$$

Hence if $\delta \in \Delta(\xi)$ is the direction defined by $\Lambda(\xi_c)$, and if $\delta' \in \Delta(\varphi(\xi))$ is the corresponding direction defined by $\Lambda(\xi_{c^p})$, then $\partial_+ H_i^{\tilde{M}}(\xi_{0,\bar{\rho}}) = \partial_+ H_{\phi(i)}^{\varphi_* \tilde{M}}(\xi_{0,\bar{\rho}^p}) - d_i$, and

$$\partial_- H_{i,\delta}^{\tilde{M}}(\xi_{0,\bar{\rho}}) = \begin{cases} \partial_- H_{\phi(i),\delta'}^{\varphi_* \tilde{M}}(\xi_{0,\bar{\rho}^p}) - d_i & \text{if } \delta \text{ is defined by } \Lambda(\xi_c) \text{ with } c = 0 \\ \partial_- H_{\phi(i),\delta'}^{\varphi_* \tilde{M}}(\xi_{0,\bar{\rho}^p}) & \text{if otherwise} \end{cases}. \quad (6.18)$$

Here the log-slopes $\partial_+ H_i^{\tilde{M}}(\xi_{0,\bar{\rho}})$ and $\partial_- H_{i,\delta}^{\tilde{M}}(\xi_{0,\bar{\rho}})$ are computed with respect to the variable $\tau = \ln(\rho)$ (cf. (2.7)), while the slopes $\partial_+ H_{\phi(i)}^{\varphi_* \tilde{M}}(\xi_{0,\bar{\rho}^p})$ and $\partial_- H_{\phi(i),\delta'}^{\varphi_* \tilde{M}}(\xi_{0,\bar{\rho}^p})$ are computed with respect to the variable $\tau' := \ln(\rho^p)$. Indeed the natural variable on $\mathcal{O}(X^p)$ is $T = \tilde{T}^p$. Moreover:

- i) There are a finite number of directions $\delta \in \Delta(\xi)$ such that $\partial_- H_{i,\delta}^{\tilde{M}}(\xi) \neq 0$ if and only if the same is true for the directions $\delta' \in \Delta(\varphi(\xi))$ such that $\partial_- H_{\phi(i),\delta'}^{\varphi_* \tilde{M}}(\varphi(\xi)) \neq 0$.
- ii) $(i, H_i^{\tilde{M}}(\xi))$ is a vertex of $NP^{\text{conv}}(\tilde{M}, \xi)$ (i.e. $i = r$ or $s_i^{\tilde{M}}(\xi) < s_{i+1}^{\tilde{M}}(\xi)$) if and only if $(\phi(i), H_{\phi(i)}^{\varphi_* \tilde{M}}(\varphi(\xi)))$ is a vertex of $NP^{\text{conv}}(\varphi_*(\tilde{M}), \varphi(\xi))$ (i.e. $\phi(i) = pr$ or $s_{\phi(i)}^{\varphi_*(\tilde{M})}(\varphi(\xi)) < s_{\phi(i)+1}^{\varphi_*(\tilde{M})}(\varphi(\xi))$).
- iii) If $K = \widehat{K}^{\text{alg}}$, and if $\xi \in X_{\text{int}}$ is not in the Shilov boundary of X , then f_i is harmonic at ξ with slopes in \mathbb{Z} . Hence $H_i^{\tilde{M}}$ is super-harmonic (resp. harmonic, has slopes in \mathbb{Z}) at $\xi := \xi_{0,\bar{\rho}}$ if and only if $H_{\phi(i)}^{\varphi_* \tilde{M}}$ is super-harmonic (resp. harmonic, has slopes in \mathbb{Z}) at $\varphi(\xi) = \xi_{0,\bar{\rho}^p}$.

The same holds for the spectral polygon since for $i \leq i_0$ one has $H_i^{\tilde{M}}(\xi) = H_i^{\tilde{M},\text{sp}}(\xi)$, and for all $j \leq \phi(i_0)$ one has $H_j^{\varphi_* \tilde{M}}(\varphi(\xi)) = H_j^{\varphi_* \tilde{M},\text{sp}}(\varphi(\xi))$.

Proof. Write $s_1^{\tilde{M}}(\xi) \leq \dots \leq s_{i_1}^{\tilde{M}}(\xi) \leq \omega|t| < s_{i_1+1}^{\tilde{M}}(\xi) \leq \dots \leq s_{i_0}^{\tilde{M}}(\xi) < \ln(\bar{\rho}) \leq s_{i_0+1}^{\tilde{M}}(\xi) \leq \dots \leq s_r^{\tilde{M}}(\xi)$. For all $i = 1, \dots, i_0$ one has $\mathcal{R}_i^{\tilde{M}} = \mathcal{R}_i^{\tilde{M},\text{sp}}$ along a conveniently small open segment containing $\xi = \xi_{0,\bar{\rho}}$ of each branch through ξ . By Prop. 6.16 for all ξ' belonging to such segments one has

$$s^{\varphi_* \tilde{M}}(\varphi(\xi')) : \overbrace{\ln(|p||t'|^{p-1}) + s_1^{\tilde{M}}(\xi') = \dots = \ln(|p||t'|^{p-1}) + s_{i_1}^{\tilde{M}}(\xi')}^{p\text{-times}} \leq \dots \quad (6.19)$$

$$\leq \overbrace{\ln(|p||t'|^{p-1}) + s_{i_1}^{\tilde{M}}(\xi') = \dots = \ln(|p||t'|^{p-1}) + s_{i_1+1}^{\tilde{M}}(\xi')}^{p\text{-times}} \leq \quad (6.20)$$

$$\leq \overbrace{\ln(\omega^p |t'|^p) = \dots = \ln(\omega^p |t'|^p)}^{(p-1)(r-i_1)\text{-times}} < p s_{i_1+1}^{\tilde{M}}(\xi') \leq \dots \leq p s_{i_0}^{\tilde{M}}(\xi') < p \ln(\bar{\rho}) \leq \dots \quad (6.21)$$

where t' is a Dwork generic point for ξ' . Hence $h_{pi}^{\varphi_* \tilde{M}}(\varphi(\xi')) = p \cdot h_i^{\tilde{M}}(\xi') + p \cdot i \cdot \ln(|p||t'|^{p-1})$ for all $i \leq i_1$. And if $i_1 < i \leq i_0$, then

$$h_{(p-1)r+i}^{\varphi_* \tilde{M}}(\varphi(\xi')) = h_{pi_1}^{\varphi_* \tilde{M}}(\varphi(\xi')) + (p-1)(r-i_1) \ln(\omega^p |t'|^p) + p s_{i_1+1}^{\tilde{M}}(\xi') + \dots + p s_i^{\tilde{M}}(\xi') \quad (6.22)$$

$$= p \cdot h_i^{\tilde{M}}(\xi') + p \cdot i_1 \cdot \ln(|p||t'|^{p-1}) + (p-1)(r-i_1) \ln(\omega^p |t'|^p) \quad (6.23)$$

$$= p \cdot h_i^{\tilde{M}}(\xi') + p \cdot r \cdot \ln(|p||t'|^{p-1}). \quad (6.24)$$

This proves (6.17). This gives $H_{i,c}^{\tilde{M}}(\rho) = H_{\phi(i),c^p}^{\varphi_* \tilde{M}}(\rho^p)^{1/p} / |f_i(\xi_{c,\rho})|$, for all $\rho \rightarrow \bar{\rho}$. So (6.18) holds. \square

7. Proof of the main Theorem 4.7

7.1 Structure of the proof

REMARK 7.1. Let $\mathcal{R}_i^M : X \rightarrow \mathbb{R}^i$ be defined by $\mathcal{R}_i^M(\xi) := (\mathcal{R}_1^M(\xi), \dots, \mathcal{R}_i^M(\xi))$. Defines analogously H_i^M, s_i^M, h_i^M . Clearly $\rho_{\mathcal{R}_i^M}(\xi) = \min_{j=1, \dots, i} \rho_{\mathcal{R}_j^M}(\xi)$, so that $\Gamma(\mathcal{R}_i^M) = \cup_{j=1, \dots, i} \Gamma(\mathcal{R}_j^M)$. Hence the finiteness of \mathcal{R}_r^M is equivalent to the finiteness of all \mathcal{R}_i^M . The same holds for H_i^M, s_i^M, h_i^M . Of course \mathcal{R}_i^M and H_i^M are the exponential of s_i^M and h_i^M respectively, so we are reduced to prove the finiteness of these latter. The functions s_i^M and h_i^M are related by the bijective map $h_i^M(\xi) = U \cdot s_i^M(\xi)$, where $U \in GL_r(\mathbb{Z})$ is the matrix $U = (u_{i,j})$ with $u_{i,j} = 1$ if $i \geq j$ and $u_{i,j} = 0$ otherwise. This proves that

$$\Gamma_i := \Gamma(\mathcal{R}_i^M) = \Gamma(H_i^M) = \Gamma(h_i^M) = \Gamma(s_i^M) \quad \text{for all } i = 1, \dots, r. \quad (7.1)$$

The aim is to apply Theorem 2.14 to the function H_i^M with respect to $\Gamma := \Gamma_{i-1}$, and a convenient finite set $\mathcal{C}_i \subseteq \Gamma_{i-1}$. The proof is an induction on i . The first step is Proposition 7.8 proving the claims for $H_1^M = \mathcal{R}_1^M$ with respect to $\Gamma_0 := \Gamma_X$, and \mathcal{C}_1 equal to the Shilov boundary.

- It easy to prove (C1),(C2),(C4), this is done in section 7.2.
- The proof of (C3) is done in section 7.4. For $\mathcal{R}_1^M = H_1^M$, (C3) is the concavity of the radius outside $\Gamma_0 = \Gamma_X$, and coincides with transfer (3.11). Now the behavior of H_i^M along a branch is not concave since it is “perturbed” by the variation of the other H_j^M , with $j < i$. The idea is then to study the locus Γ_{i-1} outside which the first $i-1$ radii are all constants. By Remark 7.1 the finiteness of all the Γ_i is equivalent to that of all the $\Gamma(H_i^M)$. So from now on we prove the finiteness of all the $\Gamma_i = \Gamma_{i-1} \cup \Gamma(H_i^M)$. Proposition 7.5 proves that H_i^M behaves as a genuine radius (of a direct factor of M) outside Γ_{i-1} . In particular H_i^M is concave outside Γ_{i-1} , and (C3) can be considered as a sort of “relative transfer” satisfied by H_i^M with respect to Γ_{i-1} .
- It remains to prove the directional finiteness (C5) and the super-harmonicity (C6) of H_i^M . They are both local properties at a point $\xi \in \Gamma(H_i^M)$, and we can assume that M is always cyclic by section 5.4.1. For this we distinguish 2 cases: $\mathcal{R}_i^M(\xi) < r(\xi)$ and $\mathcal{R}_i^M(\xi) \geq r(\xi)$.
 - * If $\mathcal{R}_i^M(\xi) < r(\xi)$ one applies Frobenius push-forward to make $\mathcal{R}_1^M, \dots, \mathcal{R}_i^M$ small, and apply Proposition 4.4 (via Thm. 5.1 ii)). This proves that H_i^M is super-harmonic at ξ , and that it has a finite number of non zero slopes $\partial_- H_{i,\delta}^M(\xi)$ (cf. section 7.3). To prove that Γ_i is directionally finite at ξ , we can forget the directions belonging to Γ_{i-1} because this last is finite by induction. Since H_i^M is concave outside Γ_{i-1} (by relative transfer (C3)) the other directions belong to Γ_i (which coincides with $\Gamma(H_i^M)$ outside Γ_{i-1}) if and only if $\partial_- H_{i,\delta}^M(\xi) < 0$, so Γ_i is directionally finite at ξ (i.e. (C6)). This is explained in Prop. 7.10.
 - * If $\mathcal{R}_i^M(\xi) \geq r(\xi)$, Lemma 7.7 essentially guarantee that $\Gamma_i = \Gamma_{i-1}$ around ξ , or, if $\xi \notin \Gamma_{i-1}$, ξ is an end point of Γ_i . So in this case the directional finiteness (C5) is easy. The super-harmonicity of H_i^M is more delicate, and it holds only outside a particular finite set \mathcal{C}_i contained in Γ_{i-1} as prescribed by Thm. 2.14. This is explained in Proposition 7.11.

7.2 Proof of Theorem 4.7 up to the finiteness and super-harmonicity.

The functions \mathcal{R}_i^M and H_i^M are insensitive to scalar extension of the ground field K , so one can always assume that the branch $\Lambda(t)$ satisfies $t \in X(K)$. So from (4.10) the function \mathcal{R}_i^M acquires immediately all the properties of $\mathcal{R}_i^{M \otimes \Omega, \text{sp}}$ along a branch. In fact, similarly to the picture (3.13), these two functions differs from an individual slope which is in both cases equal to 1 or 0. The slopes of H_i^M differs from those of $H_i^{M \otimes \Omega, \text{sp}}$ by an integer, hence ii) and iv) of Theorem 4.7 are a straightforward consequence of Thm. 5.6, and also the fact that \mathcal{R}_i^M and H_i^M verify (C2) and (C4).

REMARK 7.2. For $i = 2, \dots, r$ the functions $\mathcal{R}_{i,t}^M$ and $\mathcal{R}_{i,t}^{M, \text{sp}}$ are possibly not concave nor monotone

in $]-\infty, \rho_{t,X}[$. The picture of $\mathcal{R}_{i,t}^M$ will present then a great difference with respect to that of \mathcal{R}_1^M (cf. (3.13)). If $i = 1$, then \mathcal{R}_1^M satisfies (C3) with $\Gamma := \Gamma_X$ (cf. section 3.1).

REMARK 7.3. A part of v) and vi) of Thm. 4.7 is automatic from Thm. 5.6 if $\xi \in X_{\text{int}}$ admits, as a neighborhood, an open annulus. However we are going to reproduce entirely the general proof.

We are now reduced to prove the claims of Theorem 4.7 concerning the directional finiteness (cf. (C5)), the finiteness, the super-harmonicity, and property (C3) (cf. points iii) v), vi)).

7.3 Super-harmonicity if $\mathcal{R}_i^M(\xi) < r(\xi)$.

PROPOSITION 7.4. Let $\xi \in X_{\text{int}}$. If $\mathcal{R}_i^M(\xi) < r(\xi)$ then for all $j = 1, \dots, i$ one has $\partial_- H_{j,\delta}^M(\xi) \neq 0$ only for a finite number of directions $\delta \in \Delta(\xi)$. If $K = \widehat{K^{\text{alg}}}$, and if $\xi \in X_{\text{int}}$ is not in the Shilov boundary of X , then H_i^M is super-harmonic at ξ . If moreover $(i, h_i^M(\xi))$ is a vertex of $NP^{\text{conv}}(M, \xi)$ (i.e. $i = r$ or $s_i^M(\xi) < s_{i+1}^M(\xi)$), then $H_i^M(\xi)$ is harmonic at ξ .

Proof. One has $\mathcal{R}^M(\xi) < r(\xi) \leq \rho_{\xi, X'}$ for all $X' \subseteq X$. Hence by Prop. 5.10 all the assertions are local at ξ . So we can assume that X is a pseudo-annulus with bifurcation or punctured smooth point ξ (cf. Section 6.6), and that M is cyclic defined by an operator $\mathcal{L} = \sum_{i=0}^r g_{r-i}(T)(d/dT)^i$, $g_0 = 1$, $g_i, g_i^{-1} \in \mathcal{O}(X)$ for all i (cf. section 5.4.1). By (4.11), one has $\mathcal{R}_i^M(\xi) = \mathcal{R}_i^{M, \text{sp}}(\xi) < r(\xi)$. This equality is preserved by continuity in an open segment containing ξ of each direction $\delta \in \Delta(\xi)$. We distinguish two cases: $\mathcal{R}_i^M(\xi) < \omega \cdot r(\xi)$ and $\omega \cdot r(\xi) \leq \mathcal{R}_i^M(\xi) < r(\xi)$. In the first case by Prop. 4.3 (cf. also Prop. 4.2 and 4.4, Thm. 5.1, Cor. 5.3) for all $\delta \in \Delta(\xi)$ and all $j = 1, \dots, i$ one has

$$\partial_- \mathcal{R}_{j,\delta}^M(\xi) = \partial_- \mathcal{R}_{j,\delta}^{M, \text{sp}}(\xi) = \partial_- \mathcal{R}_{j,\delta}^{\mathcal{L}, \text{sp}}(\xi), \quad \partial_- H_{j,\delta}^M(\xi) = \partial_- H_{j,\delta}^{M, \text{sp}}(\xi) = \partial_- H_{j,\delta}^{\mathcal{L}, \text{sp}}(\xi). \quad (7.2)$$

The same equalities hold for the right slopes. Since $H_j^{\mathcal{L}, \text{sp}}$ is directional finite (cf. Prop. 4.2), then $\partial_- H_{j,\delta}^M(\xi) = 0$ up to a finite number of directions. If $\xi \in X_{\text{int}}$ is not in the Shilov boundary of X , then $H_i^{\mathcal{L}, \text{sp}}$ is super-harmonic at ξ , or harmonic if $(i, h_i^M(\xi))$ is a vertex, and so does H_i^M . If the absolute value of K extends the trivial valuation of \mathbb{Z} , then $\omega = 1$ and the proof is completed. If the absolute value of K is p -adic, then $\omega < 1$. In this case assume that $\omega r(\xi) \leq \mathcal{R}_i^M(\xi) < r(\xi)$. Up to a translation we can assume that X is obtained from an annulus $\{|T| \in [R_1, R_0]\}$ by removing some disks, and that $\xi = \xi_{0,\bar{p}}$ as in section 6.6. Then by applying several times the Frobenius push-forward we reduce the value of the radii in order to reproduce the above computations. Namely, with the notation of Proposition 6.17, let $h > 0$ be the smallest integer such that $\mathcal{R}_{\phi^h(i)}^{(\varphi_*)^h M, \text{sp}}(\varphi^h(\xi)) < \omega \cdot r(\varphi^h(\xi_{0,\bar{p}})) = \omega \bar{p}^{p^h}$, where φ^h and ϕ^h denotes the h -times iterated of φ and ϕ respectively. Then by Prop. 6.17 the slopes of H_i^M at ξ are equal to those of $H_{\phi^h(i)}^{(\varphi_*)^h M, \text{sp}}$ at $\varphi^h(\xi)$, up to the left and right slopes along $\Lambda(\xi_0)$. And H_i^M is super-harmonic at ξ if and only if so does $H_{\phi^h(i)}^{(\varphi_*)^h M, \text{sp}}$ at $\varphi^h(\xi)$. Up to restrict X^{p^h} we can assume that $(\varphi_*)^h M$ is cyclic represented by an operator $\mathcal{L}^{(h)}$ with invertible coefficients in $\mathcal{O}(X^{p^h})$. Moreover, by the choice of h , we are now in the domain of applicability of Thm. 5.1, ii), and Cor. 5.3. Hence the slopes of $H_{\phi^h(i)}^{\varphi_*^h M, \text{sp}}$ at $\varphi^h(\xi)$ are those of $\mathcal{L}^{(h)}$, and by Prop. 4.2 $H_{\phi^h(i)}^{\mathcal{L}^{(h)}, \text{sp}}$, and hence $H_{\phi^h(i)}^{\varphi_*^h M, \text{sp}}$, are super-harmonic at $\varphi^h(\xi)$. By Prop. 6.17 $(i, h_i^M(\xi))$ is a vertex if and only if $(\phi^h(i), h_{\phi^h(i)}^M(\varphi^h(\xi)))$ is a vertex, this implies the last assertion. \square

7.4 Property (C3) for H_i^M

Let $\Gamma_0 := \Gamma_X$ and $\Gamma_i := \cup_{j=1}^i \Gamma(\mathcal{R}_j^M)$. Let $D^-(t, \rho) \subset X$ be a non generic disk on which $\mathcal{R}_1^M, \dots, \mathcal{R}_{i-1}^M$ are constant i.e. $D^-(t, \rho) \cap \Gamma_{i-1} = \emptyset$. Let $b_0 := 1$, and if $i \geq 1$ let $b_i := \prod_{j=1}^i \mathcal{R}_j^M(\xi_t)$. Then $H_i^M = b_{i-1} \cdot \mathcal{R}_i^M$ over $D^-(t, \rho)$. Both functions then have the same properties on $D^-(t, \rho)$.

PROPOSITION 7.5. *If $K = \widehat{K^{\text{alg}}}$, then \mathcal{R}_i^M is either constant on $D^-(t, \rho)$ or there exists a direct sum decomposition $M \widehat{\otimes}_{\mathcal{O}(X)} \mathcal{A}_K(t, \rho) = M_1 \oplus M_2$ such that $\mathcal{R}_i^M(\xi') = \mathcal{R}_1^{M_1}(\xi')$ for all $\xi' \in D^-(t, \rho)$. In particular \mathcal{R}_i^M and H_i^M verify (C3) with respect to $\Gamma := \Gamma_{i-1}$, and they both enjoy all the properties of a genuine radius of convergence outside Γ_{i-1} . In particular $\Gamma(\mathcal{R}_i^M) \cap D^-(t, \rho) \neq \emptyset$ if and only if $\partial_- \mathcal{R}_{i,\delta}^M(\xi_{t,\rho}) < 0$ where $\delta \in \Delta(\xi_{t,\rho})$ is the direction defined by the disk.*

Proof. Since the disk is non generic we can assume $t \in X(K)$. The slopes of \mathcal{R}_i^M on the disk coincides with those of $H_i^M = b_{i-1} \mathcal{R}_i^M$. Hence, by section 7.2, \mathcal{R}_i^M verifies point iv) of Theorem 4.7 over $D^-(t, \rho)$. Assume that \mathcal{R}_i^M is non constant on $D^-(t, \rho)$. Then $\mathcal{R}_{i,t}^M(\rho') > \mathcal{R}_{i-1,t}^M(\rho')$, for all $\rho' \in]\mathcal{R}_{i-1}^M(t), \rho[$. Otherwise point iv) of Theorem 4.7 is contradicted. By Theorem 5.4 there exists a direct sum decomposition $M \widehat{\otimes} \mathcal{A}_K(t, \rho) = M_1 \oplus M_2$ of $\mathcal{A}_K(t, \rho)$ -differential modules such that for all $\rho' \in]\mathcal{R}_{i-1}^M(t), \rho[$ one has $\mathcal{R}_{k,t}^{M_1, \text{sp}}(\rho') = \mathcal{R}_{i+k-1,t}^{M_1, \text{sp}}(\rho')$, for $k = 1, \dots, r - i + 1 = \text{rank}(M_1)$, and $\mathcal{R}_{j,t}^{M_2, \text{sp}}(\rho') = \mathcal{R}_{j,t}^{M_2, \text{sp}}(\rho')$, for $j = 1, \dots, i - 1 = \text{rank}(M_2)$. By Lemma 7.6 below the radius \mathcal{R}_i^M is equal to $\mathcal{R}_1^{M_1}$ on the whole disk, and hence it enjoys its properties. Note that by (4.9) the non constancy of \mathcal{R}_i^M on $D^-(t, \rho)$ implies that $\mathcal{R}_i^M(\xi') < \rho$ for all $\xi' \in D^-(t, \rho)$, and hence $\mathcal{R}_i^M = \mathcal{R}_i^{M \widehat{\otimes} \mathcal{A}_K(t, \rho)}$ on $D^-(t, \rho)$ (cf. Def. 4.5). So $\Gamma(\mathcal{R}_i^{M \widehat{\otimes} \mathcal{A}_K(t, \rho)}) = \Gamma(\mathcal{R}_i^M) \cap D^-(t, \rho)$. Now from Lemma 2.15 this intersection is not empty if and only if $\partial_- \mathcal{R}_{i,\delta}^M(\xi_{t,\rho}) < 0$. \square

LEMMA 7.6. *For all $\xi' \in D^-(t, \rho)$ one has $\mathcal{R}_1^{M_1}(\xi') = \mathcal{R}_i^M(\xi')$.*

Proof. By Prop. 5.5 the convergence radii of M at ξ' are the union (with multiplicities) of those of M_1 and of M_2 . So it is enough to prove that for all $\xi' \in D^-(t, \rho)$ one has $\mathcal{R}_{i-1}^{M_2}(\xi') < \mathcal{R}_1^{M_1}(\xi')$. Indeed this implies by Prop. 5.5 that $\mathcal{R}_j^M(\xi') = \mathcal{R}_j^{M_2}(\xi')$ for all $j = 1, \dots, i - 1$, and $\mathcal{R}_{i+k-1}^M(\xi') = \mathcal{R}_k^{M_1}(\xi')$ for all $k = 1, \dots, r - i + 1$. Since NP^{conv} is insensitive to scalar extensions of K , one can assume that $\xi' = \xi_{t'}$ with $t' \in X(K)$. By changing the center one can assume that $t = t'$. Now $\mathcal{R}_{1,t}^{M_1}$ is log-concave with negative log-slopes along $[0, \rho[$. Then for all ρ' close enough to ρ one has $\mathcal{R}^{M_1}(\xi') = \mathcal{R}^{M_1}(\xi_t) \geq \mathcal{R}_t^{M_1}(\rho') = \mathcal{R}_{i,t}^M(\rho') > \mathcal{R}_{i-1,t}^M(\rho') = \mathcal{R}_{i-1}^M(\xi_t) = \mathcal{R}_{i-1}^M(\xi')$ as desired. \square

7.5 Finiteness and super-harmonicity (end of proof)

LEMMA 7.7. *If $\mathcal{R}_i^M(\xi) \geq r(\xi)$, then either $\xi \notin \Gamma(\mathcal{R}_i^M)$ or*

- i) *If $\xi \in \Gamma(\mathcal{R}_i^M) - \Gamma_{i-1}$, then ξ is a boundary point of $\Gamma(\mathcal{R}_i^M)$;*
- ii) *If $\xi \in \Gamma_{i-1} \cap \Gamma(\mathcal{R}_i^M)$, then $\Delta(\xi, \Gamma(\mathcal{R}_i^M)) \subseteq \Delta(\xi, \Gamma_{i-1})$.*

Proof. It is enough to prove that \mathcal{R}_i^M is constant on each non generic disk $D^-(t, \rho)$ tangent to $\xi = \xi_{t,\rho}$ such that $D^-(t, \rho) \cap \Gamma_{i-1} = \emptyset$. Up to enlarge K we can assume $K = \widehat{K^{\text{alg}}}$, and $t \in X(K)$ in order that $\rho = r(\xi)$. By Prop. 7.5 the function \mathcal{R}_i^M enjoys concavity properties on $D^-(t, \rho)$. So $\mathcal{R}_i^M(\xi_t) \geq \mathcal{R}_i^M(\xi_{t,\rho}) = \mathcal{R}_i^M(\xi) \geq r(\xi) = \rho$. Hence by (4.9) \mathcal{R}_i^M (and H_i^M) are constant on $D^-(t, \rho)$. \square

PROPOSITION 7.8. *\mathcal{R}_1^M is continuous, $\Gamma(\mathcal{R}_1^M)$ is finite and factorizes through it. If $K = \widehat{K^{\text{alg}}}$, then \mathcal{R}_1^M is super-harmonic on X . Moreover \mathcal{R}_1^M satisfies the claim vi) of Theorem 4.7.*

Proof. We can assume $K = \widehat{K^{\text{alg}}}$. In order to apply Thm. 2.14 it remains to prove directional finiteness (C5) and super-harmonicity. By Lemma 7.7 if $\mathcal{R}^M(\xi) \geq r(\xi)$, then $\Delta(\xi, \mathcal{R}^M)$ is finite in all the cases. The super-harmonicity in these cases is proved as follows. If $\xi \notin \Gamma(\mathcal{R}^M)$ there is nothing to prove. If $\xi \in \Gamma(\mathcal{R}^M) - \Gamma_X$ is a boundary point of $\Gamma(\mathcal{R}^M)$, the super-harmonicity coincides with the concavity (cf. Prop. 2.15). If $\xi \in \Gamma_X$, then $\Delta(\xi, \mathcal{R}^M) = \Delta(\xi, \Gamma_X)$, and all the directions through ξ , but those of Γ_X , are flat. Then along a branch $\Lambda(\xi_{c_i, R_i})$ of Γ_X the function $\rho \mapsto \mathcal{R}_{\xi_{c_i, R_i}}(\rho)$ is bounded by $\rho \mapsto \rho_{\xi_{c_i, R_i}, X} = \rho$ and it is equal to it at the value $\rho = \bar{\rho}$ for which

$\xi = \lambda_{\xi_{c_i}, R_i}(\bar{\rho})$. Then $\partial_+ \mathcal{R}^M(\xi) \leq 1$ and $\partial_- \mathcal{R}_\delta^M(\xi) \geq 1$ for all direction δ corresponding to a branch of Γ_X . Assume now $\mathcal{R}^M(\xi) < r(\xi)$. Apply Proposition 7.4 to prove the super-harmonicity of $\mathcal{R}^M = H_1^M$ at ξ , and that $\partial_- \mathcal{R}^M(\xi) \neq 0$ only for a finite number of directions $\delta \in \Delta(\xi)$. By Proposition 7.5 $\delta \in \Delta(\xi, \Gamma(\mathcal{R}^M)) - \Delta(\xi, \Gamma_X)$ if and only if $\partial_- \mathcal{R}^M(\xi) < 0$. This proves the directional finiteness (C5). Finally assume that $(1, h_1^M(\xi))$ is a vertex (i.e. $r = 1$ or $s_i^M(\xi) < s_{i+1}^M(\xi)$) and $\mathcal{R}^M(\xi) \neq r(\xi)$. If $\mathcal{R}^M(\xi) > r(\xi)$, then by (3.5) \mathcal{R}^M is constant on the non generic open disk $D^-(t_\xi, \mathcal{R}^M(\xi))$ that contains ξ , and there is nothing to prove. If $\mathcal{R}^M(\xi) < r(\xi)$ the harmonicity follows by Prop. 7.4. \square

COROLLARY 7.9. *Assume the M is of rank $r = 1$, and that $\xi \notin \Gamma_X$. Then ξ is a boundary point of $\Gamma(\mathcal{R}^M)$ if and only if $\mathcal{R}^M(\xi) = r(\xi)$ and $\partial_+ \mathcal{R}^M(\xi) < 0$.*

Proof. If $\mathcal{R}^M(\xi) = r(\xi)$ and $\partial_+ \mathcal{R}^M(\xi) < 0$, by Lemma 2.15, $\rho_{\mathcal{R}^M}(\xi) = r(\xi)$, hence $\xi = \delta_{\mathcal{R}^M}^M(\xi) \in \Gamma(\mathcal{R}^M)$. Now ξ lies in the boundary of $\Gamma(\mathcal{R}^M)$ by Lemma 7.7. Reciprocally by Lemma 2.15 a boundary point ξ of $\Gamma(\mathcal{R}^M)$ not in Γ_X verifies $\partial_- \mathcal{R}_\delta^M(\xi) = 0$ for all $\delta \in \Delta(\xi)$, and $\partial_+ \mathcal{R}^M(\xi) < 0$. So \mathcal{R}^M is not harmonic at ξ , and if by contrapositive $\mathcal{R}^M(\xi) \neq r(\xi)$ this contradicts point vi) of Thm. 4.7 (cf. Prop. 7.8), so one must have $\mathcal{R}^M(\xi) = r(\xi)$. \square

PROPOSITION 7.10. *If H_1^M, \dots, H_{i-1}^M are finite, then H_i^M is directionally finite.*

Proof. By assumption Γ_{i-1} is finite. If $\xi \notin \Gamma_{i-1}$, then, $\mathcal{R}_1^M, \dots, \mathcal{R}_{i-1}^M$ being constant outside Γ_{i-1} , the function H_i^M is directionally finite at ξ if and only if so does \mathcal{R}_i^M . By Prop. 7.5 the \mathcal{R}_i^M acquires the properties of a genuine radius outside Γ_{i-1} , so it is finite and super-harmonic on each non generic disk $D^-(t, \rho)$ tangent to Γ_{i-1} . Let $\xi \in \Gamma_{i-1}$. If $\mathcal{R}_i^M(\xi) \geq r(\xi)$ then one applies Lemma 7.7 to prove the directional finiteness of H_i^M since $\Gamma(H_i^M) - \Gamma_{i-1} = \Gamma(\mathcal{R}_i^M) - \Gamma_{i-1}$ (cf. Remark 7.1). If $\mathcal{R}_i^M(\xi) < r(\xi)$, by Prop. 7.5 one has $\partial_- H_{i,\delta}^M(\xi) < 0$, for all $\delta \in \Delta(\xi, \Gamma(H_i^M)) - \Delta(\xi, \Gamma_{i-1})$. Now by Prop. 7.4 there are a finite number of directions $\delta \in \Delta(\xi)$ such that $\partial_- H_{i,\delta}^M(\xi) \neq 0$. Hence $\Delta(\xi, \Gamma(H_i^M))$ is finite. \square

PROPOSITION 7.11. *If H_1^M, \dots, H_{i-1}^M satisfy Theorem 4.7, then so does H_i^M .*

Proof. It is enough to prove that H_i^M verifies claims v) and vi) of Theorem 4.7. This guarantee that H_i^M fulfill the assumptions (C1)–(C6) of Thm. 2.14 with respect to $\Gamma := \Gamma_{i-1}$ and $\mathcal{C}(H_i^M) := \mathcal{C}_i$. For this we assume $K = \widehat{K^{\text{alg}}}$. We distinguish three cases: $\mathcal{R}_i^M(\xi) < r(\xi)$, $\mathcal{R}_i^M(\xi) = r(\xi)$, and $\mathcal{R}_i^M(\xi) > r(\xi)$. Assume first that $\mathcal{R}_i^M(\xi) < r(\xi)$. By Prop. 7.4 H_i^M is super-harmonic at ξ and it also enjoys property vi) of Thm. (4.7). Assume now that $\mathcal{R}_i^M(\xi) > r(\xi)$. By (4.9) \mathcal{R}_i^M is constant on the disk non generic $D^-(t_\xi, \mathcal{R}_i^M(\xi))$ that contains ξ , and hence $H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M(\xi)$ is super-harmonic at ξ if and only if so does H_{i-1}^M . This happens if $\xi \notin \mathcal{C}_{i-1} \subseteq \mathcal{C}_i$ so v) of Thm. 4.7 is fulfilled. We now check vi). Assume that $r(\xi) \notin \{\mathcal{R}_j^M(\xi)\}_{j=1, \dots, i}$ and that $(i, h_i^M(\xi))$ is a vertex. Let i_0 be the largest value of j such that $\mathcal{R}_j^M(\xi) < r(\xi)$, or $i_0 = 0$ if $\mathcal{R}_1^M(\xi) > r(\xi)$. By Prop. 7.4 $H_{i_0}^M$ is harmonic at ξ . Since all the functions $\mathcal{R}_{i_0+1}^M, \dots, \mathcal{R}_i^M$ are flat at ξ , then $H_i^M = H_{i_0}^M \cdot \prod_{j=i_0+1}^i \mathcal{R}_j^M$ is harmonic at ξ . This concludes the case $\mathcal{R}^M(\xi) > r(\xi)$. Assume now that $\mathcal{R}_i^M(\xi) = r(\xi)$. In this case we only have to check property v) of Thm. 4.7. We analyze the possible cases. If $\xi \notin \Gamma(H_i^M)$, then H_i^M is flat at ξ , and hence harmonic. If $\xi \notin \Gamma_{i-1}$, then H_{i-1}^M is flat, and \mathcal{R}_i^M enjoys the properties of a radius (cf. Prop. 7.5) and it is super harmonic at ξ . So $H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M$ is super-harmonic at ξ . If $\xi \notin \Gamma(\mathcal{R}_i^M)$, then \mathcal{R}_i^M is flat at ξ , and by induction H_{i-1}^M is super-harmonic outside $\mathcal{C}_{i-1} \subseteq \mathcal{C}_i$. So if $\xi \notin \mathcal{C}_i$, then $H_i^M = H_{i-1}^M \cdot \mathcal{R}_i^M$ is super-harmonic at ξ . Assume then that $\mathcal{R}_i^M(\xi) = r(\xi)$, and that $\xi \in \Gamma(H_i^M) \cap \Gamma_{i-1} \cap \Gamma(\mathcal{R}_i^M)$. We have to prove that H_i^M is super-harmonic outside \mathcal{C}_{i-1} and of the boundary of $\Gamma(\mathcal{R}_i^M)$. Since $\xi \notin \mathcal{C}_{i-1}$, then H_{i-1}^M is super-harmonic at ξ , so it is enough to prove that \mathcal{R}_i^M is super-harmonic at ξ . By Lemma 7.7 we know that $\Delta(\xi, \Gamma(\mathcal{R}_i^M)) \subseteq \Delta(\xi, \Gamma_{i-1})$. Since ξ is not in the boundary of $\Gamma(\mathcal{R}_i^M)$, then $\Delta(\xi, \Gamma(\mathcal{R}_i^M))$ is not empty. We now prove that for

all $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M))$ one has

$$\partial_- \mathcal{R}_{i,\delta}^M(\xi) \geq 1 \geq \partial_+ \mathcal{R}_i^M(\xi). \quad (7.3)$$

And hence that \mathcal{R}_i^M is super-harmonic at ξ . Let $\Lambda(\xi_t)$, $t \in K$, be a representative branch of a direction $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M))$, and let $\xi = \xi_{t,\bar{\rho}}$. If $\delta \in \Delta(\xi, \Gamma_X)$, then by definition $\mathcal{R}_i^M(\xi') \leq \rho_{\xi',X} = r(\xi')$ for all $\xi' \in \Gamma_X$, so that $\mathcal{R}_{i,t}^M(\rho) \leq \rho = r(\xi_{t,\rho})$ around $\bar{\rho}$, and $\mathcal{R}_i^M(\xi) = \mathcal{R}_{i,t}^M(\bar{\rho}) = \bar{\rho} = r(\xi)$. This implies (7.3). Assume now that $\delta \in \Delta(\xi, \Gamma(\mathcal{R}_i^M)) - \Delta(\xi, \Gamma_X)$. Then one must have $\mathcal{R}_i^M(\xi_t) < \bar{\rho}$ otherwise \mathcal{R}_i^M is constant on $D^-(t, \bar{\rho})$ and $\delta \notin \Delta(\xi, \Gamma(\mathcal{R}_i^M))$. By (4.10) for all $\rho \geq \mathcal{R}_i^M(\xi_t)$ one has $\mathcal{R}_{i,t}^M(\rho) = \mathcal{R}_{i,t}^{M,\text{sp}}(\rho) \leq \rho = r(\xi_{t,\rho})$, and as above $\mathcal{R}_i^M(\xi) = \mathcal{R}_{i,t}^M(\bar{\rho}) = \bar{\rho} = r(\xi)$. This implies (7.3). \square

The above properties imply the finiteness of each H_i^M (and hence of \mathcal{R}_i^M , s_i^M , h_i^M) by applying inductively Thm. 2.14 to $\mathcal{R} := H_i^M$, $\Gamma := \Gamma_{i-1}$, and $\mathcal{C}(\mathcal{R}) := \mathcal{C}_i$. This ends the proof of Thm. 4.7.

Proof of Corollary 4.8. By translation we can assume $t = 0$. Let \mathcal{O} be equal to one of $\mathcal{H}_K(0, I)$, $\mathcal{B}_K(0, I)$, $\mathcal{A}_K(0, I)$. Almost all assertions can be proved from Thm. 4.7 by restriction to a sub-annulus (resp. sub-disk) $\{|T| \in J\}$, with J compact (resp. $0 \in J$). The unique assertion that remains to prove are the global finiteness of each H_i^M , and the fact that along the branch $\lambda_0(I) := \{\xi_{0,\rho}\}_{\rho \in I}$ each H_i^M has a finite number of breaks. By Cor. 2.21, these two assertions are in fact equivalent, and this proves iii). This also proves the last assertion. Namely assume that, for $i \leq r$, all H_1^M, \dots, H_i^M have a finite number of breaks along $\lambda_0(I)$. Then $\mathcal{R}_1^M = H_1^M$ is finite by Cor. 2.21, and an induction on $k \leq i$ shows that $\Gamma(H_k^M)$ is contained in $A(0, J_k) \cup \lambda_0(I)$ for some compact J_k . Indeed each new branch generates by super-harmonicity a break along $\lambda_0(I)$. We now prove i) and ii). If $\mathcal{O} = \mathcal{H}_K(0, I)$ or if $\mathcal{O} = \mathcal{B}_K(0, I)$ with K discretely valued, then \mathcal{R}_i^M and H_i^M always have a finite number of breaks along $\lambda_0(I)$ by [Ked10b, Thm. 11.3.2, Remark 11.3.4]. The proof ends here, but for the convenience of the reader we now show why this is true. We assume that I is open, since otherwise the result is Thm. 4.7. For all i one has $\mathcal{R}_i^M = \mathcal{R}_i^{M,\text{sp}}$ along $\lambda_0(I)$. Let $L_i := \lim_{\rho \rightarrow s^-} \mathcal{R}_i^{M,\text{sp}}(\xi_{0,\rho})$, where $s := \sup(I)$. The limit exists in $[0, s]$ since $H_i^{M,\text{sp}}$ is concave along I , for all i . If $L_1 = s$,¹⁹ then the sequence of slopes of $\mathcal{R}_1^{M,\text{sp}}$ is decreasing (by concavity), contained in $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{r}\mathbb{Z}$, and lower bounded by 1 (by definition of spectral radius). So the sequence of slopes is constant for $\rho \rightarrow s^-$: there is a “last slope” (this argument is due to Christol-Mebkhout [CM02]). So \mathcal{R}_1^M is log-linear outside some compact $J_1 \subseteq I$, and hence finite by Cor. 2.21. Since H_2^M is log-concave, then \mathcal{R}_2^M is concave outside J_1 , so the same argument proves that \mathcal{R}_2^M is log-linear outside some $J_2 \supseteq J_1$, and hence finite by Cor. 2.21. By induction all \mathcal{R}_i^M are finite. Assume now that $L_1 < s$. Let i_0 be the larger value of i such that $L_i < s$. We perform Frobenius push-forward to have $L_{i_0} < \omega s$. By continuity there exists $\varepsilon > 0$ such that $\mathcal{R}_i^M(\xi_{0,\rho}) < \omega \rho$ for all $\rho \in [s - \varepsilon, s[$. To prove that the number of breaks of \mathcal{R}_1^M is finite on $[s - \varepsilon, s[$ one has to perform a global push-forward over $[s - \varepsilon, s[$ in order to control simultaneously all the $\{\mathcal{R}_i^M(\xi_{0,\rho})\}_{\rho \in [s - \varepsilon, s[}$. So one argues as in section 6.3 replacing $\mathcal{H}(\xi)$ by $\mathcal{H}_K([s - \varepsilon, s])$ or $\mathcal{B}_K([s - \varepsilon, s])$. One has the same results as in section 6.5. One sees from (6.12), that if the coefficients of $G(\tilde{T})$ are bounded (resp. analytic elements), then so does $\varphi_*(G)(T)$. Indeed the sequence $\{a_i\}_i$ of the Taylor coefficients of the entries $h_{i,j}(T) = \sum_{i \in \mathbb{Z}} a_i T^i$ of $\varphi_*(G)(T)$ are obtained as sub-sequences of those of $G(\tilde{T})$. Now we perform a base change in the fraction field of \mathcal{O} , and possibly restrict the annulus, to find a differential operator that again has bounded coefficients (resp. analytic elements). Now bounded functions (resp. analytic elements) have a finite number of zeros²⁰ so the Newton polygon of the operator has a finite number of slopes. This proves that $\mathcal{R}_1^M, \dots, \mathcal{R}_{i_0}^M$ have a finite number of breaks along $\lambda_0(I)$, and hence that they are finite by Cor. 2.21. For $i \geq i_0 + 1$ one proceeds inductively using the above argument of Christol-Mebkhout, to prove the finiteness of \mathcal{R}_i^M . \square

¹⁹This condition is called solvability in [CM02].

²⁰Bounded function have a finite number of zeros if and only if K has a discrete valuation [Chr12].

8. S -skeleton

In [Bal10] and in [PP12] one considers a slight modified definition of i -th radii depending on a given skeleton Γ as follows. Let $\mathfrak{f} \subset X_{\text{int}}$ be a finite subset and let $\Gamma = \text{Sat}(\mathfrak{f})$ be a finite and branch-closed saturated subset of the affinoid domain $X \subseteq \mathbb{A}_K^{1,\text{an}}$. The union of all bifurcation, punctured smooth, and boundary points of Γ constitute the so called *(weak) triangulation* S of X (cf. [PP12]). The data of $\Gamma \cup \Gamma_X$, or equivalently of S , defines univocally a covering of X by (possibly not K -rational) closed annuli and closed disks such that

- i) the union of their skeletons equals $\Gamma \cup \Gamma_X$,
- ii) the union of the boundary points of their skeletons is S .

Imitating section 2 define the S -constancy radius of an arbitrary function $\mathcal{R} : X \rightarrow \mathcal{T}$ as

$$\rho_{\mathcal{R};S}(\xi) := \min(\rho_{\mathcal{R}}(\xi), \rho_{\Gamma}(\xi)) , \quad (8.1)$$

and the S -skeleton $\Gamma(\mathcal{R};S)$ of \mathcal{R} as the image of the map $\delta_{\mathcal{R};S} : X \rightarrow X$ defined by $\delta_{\mathcal{R};S}(\xi) := \lambda_{\xi}(\rho_{\mathcal{R};S}(\xi))$. From (8.1) one immediately has $\Gamma(\mathcal{R};S) = \Gamma(\mathcal{R}) \cup \Gamma$. So the finiteness of $\Gamma(\mathcal{R})$ is equivalent to that of $\Gamma(\mathcal{R};S)$.

Now let M be a differential module over $\mathcal{O}(X)$ of rank r . Define $\mathcal{R}_{S,i}^M(\xi)$ as the larger value of $\rho \leq \rho_{\Gamma}(\xi)$ such that there exists $\Omega \in E(K)$, and a Dwork generic point t for ξ , in order that M has at least $r - i + 1$ linearly independent solution with values in $\mathcal{A}_{\Omega}(t, \rho)$. One sees that $\mathcal{R}_{S,i}^M(\xi) := \min(\mathcal{R}_i^M(\xi), \rho_{\Gamma}(\xi))$, so that by Remarks 2.4 and 2.3 one has²¹ $\rho_{\mathcal{R}_{S,i}^M}(\xi) \geq \min(\rho_{\Gamma \cup \Gamma_X}(\xi), \rho_{\mathcal{R}_i^M}(\xi))$ and $\Gamma(\mathcal{R}_{S,i}^M) \subseteq \Gamma(\mathcal{R}_i^M) \cup \Gamma$. Its finiteness and branch continuity are clear by Thm. 4.7, and hence its continuity. On the other hand its S -skeleton is $\Gamma(\mathcal{R}_{S,i}^M, S) = \Gamma(\mathcal{R}_i^M, S) = \Gamma(\mathcal{R}_i^M) \cup \Gamma$. Both \mathcal{R}_i^M and $\mathcal{R}_{S,i}^M$ are continuous and factorizes through $\Gamma(\mathcal{R}_i^M, S)$.

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²¹More precisely by (4.9) and since $\rho_{\Gamma}(\lambda_{\xi}(\rho)) = \max(\rho_{\Gamma}(\xi), \rho)$, one has $\rho_{\mathcal{R}_{S,i}^M}(\xi) = \begin{cases} \rho_{\mathcal{R}_i^M}(\xi) & \text{if } \mathcal{R}_i^M(\xi) \leq \rho_{\Gamma}(\xi) \\ \rho_{\Gamma}(\xi) & \text{if } \mathcal{R}_i^M(\xi) > \rho_{\Gamma}(\xi) \end{cases}$.

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